Free extensions

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For general algebraic reasons, every bounded distributive lattice L admits a universal homomorphism $f\colon L \to B$ with a Boolean algebra as a codomain. (In other words, the forgetful functor from the category of Boolean algebras to the category of bounded distributive lattices has a left adjoint.) Furthermore, the universal homomorphism $f\colon L \to B$ is injective, and so it is often referred to as the *free Boolean extension* of L [1, p. 97]. We focus our attention on the following known fact:

If A is a Boolean algebra and $g\colon L \hookrightarrow A$ is an injective homomorphism of bounded distributive lattices such that the image of g generates A as a Boolean algebra, then g is a free Boolean extension.

In short:

$$\text{injective} + \text{generating} \Rightarrow \text{free}.$$

A similar fact occurs for other algebraic structures. For example: every commutative monoid M admits a universal homomorphism $f\colon M \to G$ to an Abelian group G (the so-called Grothendieck group), which furthermore is an injective map if M is cancellative. Moreover,

If H is an Abelian group and $g\colon M \hookrightarrow H$ is an injective monoid homomorphism such that the image of g generates H as a group, then g is universal among the homomorphisms from M to the monoid reducts of Abelian groups.

Again,

$$\text{injective} + \text{generating} \Rightarrow \text{free}.$$

In contrast, the corresponding property for arbitrary (i.e., not necessarily commutative) groups and monoids does not hold: the free monoid over a three-element set can be embedded in non-isomorphic ways into the free group over a three-element set and into the free group over a two-element set.

The purpose of this work is to characterize the situations in which the property "injective + generating ⇒ free" (which we call "free extension property") holds.

Let us fix a functional language $\mathcal{L}_+$, a sublanguage $\mathcal{L}_- \subseteq \mathcal{L}_+$, an SP-class V_+ of algebras for $\mathcal{L}_+$ and an SP-class V_- of algebras for $\mathcal{L}_-$ that contains all $\mathcal{L}_-$-reducts of V_+. For example: $\mathcal{V}_+$ and $\mathcal{V}_-$ can be taken as the classes of Abelian groups and cancellative commutative monoids, respectively, or as the classes of Boolean algebras and bounded distributive lattices, or as the classes of groups and monoids.

Definition (Free extension property)**.** We say that V_+ *has the free extension property over* V_- if, for all $\mathbf{A} \in \mathsf{V}_-$, all $\mathbf{B}, \mathbf{C} \in \mathsf{V}_+$, all $\mathcal{L}_-$-homomorphisms $f\colon \mathbf{A} \hookrightarrow \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{C}$ such

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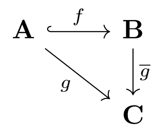
that f is injective and the image of f $\mathcal{L}_+$-generates $\mathbf{B}$, there is a unique $\mathcal{L}_+$-homomorphism $\overline{g}\colon \mathbf{B} \to \mathbf{C}$ making the following diagram commute.



We will connect the free extension property with the following property.

Definition (Expressibility in V_- of equations in V_+)**.** We say that *equations in* V_+ *are expressible in* V_- if, for every pair $(\sigma(x_1,\dots,x_n), \rho(x_1,\dots,x_n))$ of terms in V_+ there is a family $\{(\alpha_i(x_1,\dots,x_n), \beta_i(x_1,\dots,x_n)) \mid i \in I\}$ of pairs of terms in V_- such that, for every $\mathbf{A} \in \mathsf{V}_+$ and every n-tuple $\mathbf{a} = (a_1,\dots,a_n)$ of elements of A, we have

$$[\![\sigma]\!]^{\mathbf{A}}(\mathbf{a}) = [\![\tau]\!]^{\mathbf{A}}(\mathbf{a}) \iff \text{for all } i \in I \ [\![\alpha_i]\!]^{\mathbf{A}}(\mathbf{a}) = [\![\beta_i]\!]^{\mathbf{A}}(\mathbf{a}).$$

The following is our main result.

Theorem. V_+ *has the free extension property over* V_- *if and only if equations in* V_+ *are expressible in* V_-.

Some examples where the two equivalent conditions are satisfied are: Boolean algebras and bounded distributive lattices, Abelian groups and commutative monoids, Abelian ℓ-groups and commutative ℓ-monoids, MV-algebras and their positive subreducts, rational vector spaces and Abelian groups. For example, the equation $x - y = w - z$ in the language of Abelian groups is expressible in the language of monoids as $x + z = y + w$, and the equation $x = y \wedge \neg z$ in the language of Boolean algebras is expressible as the system of equations $\{x \wedge z = 0,\ x \vee z = y \vee z\}$ in the language of bounded distributive lattices.

Some examples where the two properties are not satisfied are: groups and monoids (where the equation $x = yz^{-1}y$ cannot be expressed by monoid equations), modal algebras and positive modal algebras (where the equation $x = \Diamond(y \wedge \neg z)$ cannot be expressed by equations in the positive language), complex vector spaces and real vector spaces (where the equation $x = iy$ cannot be expressed by equations in the language of real vector spaces).

Our result holds also for SP-classes of possibly infinitary algebras in a possibly large signature, provided we require the existence of free algebras (which is no longer guaranteed when the signature is large). Moreover, $\mathcal{L}_-$ does not need to be contained in $\mathcal{L}_+$; a term-interpretation of $\mathcal{L}_-$ into $\mathcal{L}_+$ (or, equivalently, a functor $\mathsf{V}_+ \to \mathsf{V}_-$ commuting with the underlying functors to Set) suffices.

References

[1] R. Balbes and P. Dwinger. *Distributive lattices.* University of Missouri Press, Columbia, Mo., 1974.