Stone duality for locally finitely residual algebras with a near unanimity term

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Keimel & Werner [6] obtained a duality for classes of the form $\mathbb{ISP}(\mathbf{L})$ where $\mathbf{L}$ is a finite quasi-primal algebra. Davey & Werner [5] extended this result to the case where $\mathbf{L}$ is a finite algebra with a near unanimity term. Our main result is an extension of these dualities to the case of a possibly infinite algebra $\mathbf{L}$ (subject to certain hypothesis), although in this case we have to restrict to a subclass of $\mathbb{ISP}(\mathbf{L})$.

1 Description of the main result

For a (possibly infinite) hereditarily finitely subdirectly irreducible algebra $\mathbf{L}$ with a near unanimity term, we provide a duality for the class of algebras $\mathbf{A}$ in $\mathbb{ISP}(\mathbf{L})$ *with finitely* $\mathbf{L}$*-valued elements*, i.e. such that for each $a \in \mathbf{A}$ the set $\{h(a) \mid h\colon \mathbf{A} \to \mathbf{L} \text{ homomorphism}\}$ is finite. We note that, if $\mathbf{L}$ is finite, any algebra in $\mathbb{ISP}(\mathbf{L})$ has finitely $\mathbf{L}$-valued elements, and so in this case we obtain a duality for the whole class $\mathbb{ISP}(\mathbf{L})$, as in [5].

In this duality, to each algebra $\mathbf{A}$ is associated a structured set $\mathbb{X}$ in a way such that each element of $\mathbf{A}$ is represented by a function from $\mathbb{X}$ to $\mathbf{A}$ with finite image, and the operations are computed pointwise.

For the sake of simplicity, we present our result in the case where, for each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$ is the inclusion.

For a natural number k, we let $I \subseteq_k X$ stand for "I is a subset of X with cardinality less than or equal to k".

The following structures will be shown to be dual to algebras in $\mathbb{ISP}(\mathbf{L})$ with finitely $\mathbf{L}$-valued elements.

Definition 1 (Priestley $\mathbf{L}$-spaces)**.** *Let $k \geq 2$, and let $\mathbf{L}$ be a hereditarily finitely subdirectly irreducible algebra $\mathbf{L}$ with a $(k+1)$-near unanimity term. Suppose that, for each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$. A* Priestley $\mathbf{L}$-space *consists of a Stone space X, and, for each $I \subseteq_k X$, of a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$ with the following properties.*

1. *(Global extension) For every $I \subseteq_k X$ and every $f \in \mathbf{A}_I$ there is a continuous function $g\colon X \to \mathbf{L}$ (where $\mathbf{L}$ is equipped with the discrete topology) such that $g|_I = f$ and, for every $J \subseteq_k X$, $g|_J \in \mathbf{A}_J$.*
2. *(Separation) For all distinct $x, y \in X$, there is $f \in \mathbf{A}_{\{x,y\}}$ such that $f(x) \neq f(y)$.*

Definition 2 (Morphism of Priestley $\mathbf{L}$-spaces)**.** *A morphism of Priestley $\mathbf{L}$-spaces f from $(X, (\mathbf{A}_I)_{I \subseteq_k X})$ to $(X', (\mathbf{A}'_I)_{I \subseteq_k X'})$ is a continuous function $f\colon X \to X'$ such that, for all $I \subseteq_k X$ and all $g \in \mathbf{A}'_{f[I]}$, $(g \circ f)|_I \in \mathbf{A}_I$.*

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The following is our main result.

Theorem 3. *Let $k \geq 2$, and let $\mathbf{L}$ be a hereditarily finitely subdirectly irreducible algebra $\mathbf{L}$ with a $(k+1)$-near unanimity term. Suppose that, for each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$. The category of algebras in $\mathbb{ISP}(\mathbf{L})$ with finitely $\mathbf{L}$-valued elements is dually equivalent to the category of Priestley $\mathbf{L}$-spaces.*

2 Applications

As an application of our result, we obtain a representation of the *positive MV-algebras* (i.e. $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras [2]) with finitely $[0,1]$-valued elements; here $[0,1]$ denotes the $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-reduct of the standard MV-algebra $[0,1]$. This was the main motivation for our investigation.

For $\mathbf{L}$ the standard MV-algebra $[0,1]$, our main result specializes to the duality for weakly locally finite MV-algebras of [4], which in turn generalized the duality for locally finite MV-algebras in [3]. This shows that the dualities for MV-algebras in [3, 4] are special instances of a result in universal algebra.

A further application of our main result is a simple description, via duality, of the free MV-extension (introduced in [1]) of all positive MV-algebras with finitely $[0,1]$-valued elements.

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