Vietoris endofunctor for closed relations

Marco Abbadini*

University of Birmingham, University Rd W, Birmingham B15 2TT
m.abbadini@bham.ac.uk

Taking the Vietoris hyperspace $\mathbb{V}(X)$ of a compact Hausdorff space X defines an endofunctor $\mathbb{V}$ on the category KHaus of compact Hausdorff spaces and continuous maps. On morphisms, a continuous function $f\colon X \to Y$ is mapped to the function $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ that maps a closed subset F of X to the image $f[F]$ of F under f.

The larger category $\mathsf{KHaus}^{\mathsf{R}}$ of compact Hausdorff spaces and closed relations has been investigated in various works [11, 8, 10, 5, 1]. One appealing feature of $\mathsf{KHaus}^{\mathsf{R}}$ is that it is self-dual. We generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$. For a closed relation $R \subseteq X \times Y$, we define $\mathbb{V}^{\mathsf{R}}(R)$ by generalizing the well-known Egli-Milner order: for all closed subsets $F \subseteq X$ and $G \subseteq Y$, we set

$$F \ \mathbb{V}^{\mathsf{R}}(R)\ G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R-image of F in Y and $R^{-1}[G]$ is the R-preimage of G in X. We show that this defines an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ that restricts to the Vietoris endofunctor $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ and commutes with the self-duality of $\mathsf{KHaus}^{\mathsf{R}}$.

De Vries duality [7] is a categorical dual equivalence for KHaus which associates with each compact Hausdorff space X the boolean algebra $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. This yields a duality between KHaus and the category DeV of *de Vries algebras*, i.e. pairs $(B, \prec)$ where B is a complete boolean algebra and $\prec$ is a proximity relation on B. A direct pointfree construction of the endofunctor $\mathsf{DeV} \to \mathsf{DeV}$ dual to $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ remained an open problem [4, p. 375]. We resolve this problem as follows.

In [1], we extended de Vries duality to $\mathsf{KHaus}^{\mathsf{R}}$. Let $\mathsf{Stone}^{\mathsf{R}}$ be the full subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. Stone duality extends to an equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and the category BA^{S} with boolean algebras as objects and subordination relations as morphisms [6, 9, 1]. This yields an equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and a category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms generalizing the axioms of an $\mathsf{S5}$-modality. Because of this connection, we termed the pairs (B, S) $\mathsf{S5}$-*subordination algebras* and denoted the resulting category by $\mathsf{SubS5}^{\mathsf{S}}$ [1]. The inclusion $\mathsf{DeV}^{\mathsf{S}} \hookrightarrow \mathsf{SubS5}^{\mathsf{S}}$ of the full subcategory $\mathsf{DeV}^{\mathsf{S}}$ consisting of de Vries algebras into $\mathsf{SubS5}^{\mathsf{S}}$ is an equivalence, with quasi-inverse obtained by generalizing the MacNeille completion to $\mathsf{S5}$-subordination algebras [2].

In [12], the endofunctor $\mathbb{K}$ on boolean algebras dual to the Vietoris endofunctor $\mathbb{V}$ on Stone spaces was defined. We lift $\mathbb{K}$ to an endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S} equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{Stone}^{\mathsf{R}}$. Finally, we lift $\mathbb{K}^{\mathsf{S}}$ to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. Composing it with the MacNeille completion yields an endofunctor on $\mathsf{DeV}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$. This solves the problem mentioned above in the category $\mathsf{SubS5}^{\mathsf{S}}$, in its full subcategory $\mathsf{DeV}^{\mathsf{S}}$, and finally in DeV via a duality between DeV and a wide subcategory of $\mathsf{DeV}^{\mathsf{S}}$.

References

*This is joint work with Guram Bezhanishvili (New Mexico State University) and Luca Carai (University of Milan).

[1] M. Abbadini, G. Bezhanishvili, and L. Carai. A generalization of de Vries duality to closed relations between compact Hausdorff spaces. *Topology Appl.*, 337:Paper No. 108641, 2023.

[2] M. Abbadini, G. Bezhanishvili, and L. Carai. MacNeille completions of subordination algebras. *Cah. Topol. Géom. Différ. Catég.*, 65:151–199, 2024.

[3] M. Abbadini, G. Bezhanishvili, and L. Carai. Vietoris endofunctor for closed relations and its de Vries dual. *Topology Proc.*, 2024. To appear. Preprint at arXiv:2308.16823.

[4] G. Bezhanishvili, N. Bezhanishvili, and J. Harding. Modal operators on compact regular frames and de Vries algebras. *Appl. Categ. Structures*, 23(3):365–379, 2015.

[5] G. Bezhanishvili, D. Gabelaia, J. Harding, and M. Jibladze. Compact Hausdorff spaces with relations and Gleason spaces. *Appl. Categ. Structures*, 27(6):663–686, 2019.

[6] S. A. Celani. Quasi-semi-homomorphisms and generalized proximity relations between Boolean algebras. *Miskolc Math. Notes*, 19(1):171–189, 2018.

[7] H. de Vries. *Compact spaces and compactifications. An algebraic approach.* PhD thesis, University of Amsterdam, 1962.

[8] A. Jung, M. Kegelmann, and M. A. Moshier. Stably compact spaces and closed relations. In *MFPS 2001. Papers from the 17th Conference on the Mathematical Foundations of Programming Semantics, Aarhus University, Aarhus, Denmark, May 23–26, 2001*, pages 209–231. Amsterdam: Elsevier, 2001.

[9] A. Kurz, A. Moshier, and A. Jung. Stone duality for relations. In A. Palmigiano and M. Sadrzadeh, editors, *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, pages 159–215. Springer International Publishing, Cham, 2023.

[10] M. A. Moshier. On the relationship between compact regularity and Gentzen's cut rule. *Theor. Comput. Sci.*, 316(1-3):113–136, 2004.

[11] C. F. Townsend. *Preframe techniques in constructive locale theory.* PhD thesis, Imperial College London (University of London), 1996.

[12] Y. Venema and J. Vosmaer. Modal logic and the Vietoris functor. In G. Bezhanishvili, editor, *Leo Esakia on duality in modal and intuitionistic logics*, volume 4 of *Outst. Contrib. Log.*, pages 119–153. Springer, Dordrecht, 2014.