

Vietoris endofunctor for closed relations and its de Vries dual*

Marco Abbadini^{1†}, Guram Bezhanishvili², and Luca Carai³

¹ University of Birmingham, Birmingham, U.K.
marco.abbadini.uni@gmail.com

² New Mexico State University, Las Cruces, New Mexico, U.S.A.
guram@nmsu.edu

³ Università degli Studi di Milano, Milan, Italy
luca.carai.uni@gmail.com

Abstract

We generalize the Vietoris endofunctor to the category of compact Hausdorff spaces and closed relations and describe the dual endofunctor on the category of de Vries algebras and subordinations.

Taking the Vietoris hyperspace $\mathbb{V}(X)$ of a compact Hausdorff space X defines an endofunctor \mathbb{V} on the category \mathbf{KHaus} of compact Hausdorff spaces and continuous functions. On morphisms, a continuous function $f: X \rightarrow Y$ is mapped to the function $\mathbb{V}(f): \mathbb{V}(X) \rightarrow \mathbb{V}(Y)$ which maps a closed subset F of X to the image $f[F]$ of F under f .

The larger category \mathbf{KHaus}^R of compact Hausdorff spaces and closed relations has been investigated in various works [11, 8, 10, 5, 1]. One appealing feature of \mathbf{KHaus}^R is that it is self-dual. We generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^R: \mathbf{KHaus}^R \rightarrow \mathbf{KHaus}^R$. For a subordination $R \subseteq X \times Y$, we define $\mathbb{V}^R(R)$ by generalizing the well-known Egli-Milner order: for all closed subsets $F \subseteq X$ and $G \subseteq Y$,

$$F \mathbb{V}^R(R) G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R -image of F in Y and $R^{-1}[G]$ is the R -preimage of G in X . We show that this defines an endofunctor $\mathbb{V}^R: \mathbf{KHaus}^R \rightarrow \mathbf{KHaus}^R$ which restricts to the Vietoris endofunctor $\mathbb{V}: \mathbf{KHaus} \rightarrow \mathbf{KHaus}$ and commutes with the self-duality of \mathbf{KHaus}^R .

De Vries duality [7] is a duality for \mathbf{KHaus} which associates with each compact Hausdorff space X the boolean algebra $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ iff $\text{cl}(U) \subseteq V$. This yields a duality between \mathbf{KHaus} and the category \mathbf{DeV} of *de Vries algebras*, i.e. pairs (B, \prec) where B is a complete boolean algebra and \prec is a proximity relation on B . A direct pointfree construction of the endofunctor $\mathbf{DeV} \rightarrow \mathbf{DeV}$ dual to $\mathbb{V}: \mathbf{KHaus} \rightarrow \mathbf{KHaus}$ remained an open problem [4, p. 375]. We resolve this problem as follows.

In [1] we extended de Vries duality to \mathbf{KHaus}^R . Let \mathbf{Stone}^R be the full subcategory of \mathbf{KHaus}^R consisting of Stone spaces. Stone duality extends to an equivalence between \mathbf{Stone}^R and the category \mathbf{BA}^S with boolean algebras as objects and subordination relations as morphisms [6, 9, 1]. This yields an equivalence between \mathbf{KHaus}^R and a category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms generalizing the axioms of an $\mathbf{S5}$ -modality. Because of this connection, we termed the pairs (B, S) *S5-subordination algebras* and denoted the resulting category by $\mathbf{SubS5}^S$ [1]. The inclusion $\mathbf{DeV}^S \hookrightarrow \mathbf{SubS5}^S$ of the full subcategory \mathbf{DeV}^S consisting of de Vries algebras

*This presentation is based on [3].

†Presenter

is an equivalence, with quasi-inverse obtained by generalizing the MacNeille completion to $S5$ -subordination algebras [2].

In [12], the endofunctor \mathbb{K} on boolean algebras dual to the Vietoris endofunctor \mathbb{V} on Stone spaces was defined. We lift \mathbb{K} to an endofunctor \mathbb{K}^S on \mathbf{BA}^S equivalent to \mathbb{V}^R on \mathbf{Stone}^R . Finally, we lift \mathbb{K}^S to an endofunctor on $\mathbf{SubS5}^S$ equivalent to \mathbb{V}^R on \mathbf{KHaus}^R . Composing it with the MacNeille completion yields an endofunctor on \mathbf{DeV}^S equivalent to \mathbb{V}^R . This solves the problem mentioned above in the category $\mathbf{SubS5}^S$, in its full subcategory \mathbf{DeV}^S , and finally in \mathbf{DeV} via a duality between \mathbf{DeV} and a wide subcategory of \mathbf{DeV}^S .

References

- [1] M. Abbadini, G. Bezhanishvili, and L. Carai. A generalization of de Vries duality to closed relations between compact Hausdorff spaces. *Topology Appl.*, 337:Paper No. 108641, 2023.
- [2] M. Abbadini, G. Bezhanishvili, and L. Carai. MacNeille completions of subordination algebras. *Cah. Topol. Géom. Différ. Catég.*, 65:151–199, 2024.
- [3] M. Abbadini, G. Bezhanishvili, and L. Carai. Vietoris endofunctor for closed relations and its de Vries dual. *Topology Proc.*, 2024. To appear. Preprint at [arXiv:2308.16823](https://arxiv.org/abs/2308.16823).
- [4] G. Bezhanishvili, N. Bezhanishvili, and J. Harding. Modal operators on compact regular frames and de Vries algebras. *Appl. Categ. Structures*, 23(3):365–379, 2015.
- [5] G. Bezhanishvili, D. Gabelaia, J. Harding, and M. Jibladze. Compact Hausdorff spaces with relations and Gleason spaces. *Appl. Categ. Structures*, 27(6):663–686, 2019.
- [6] S. A. Celani. Quasi-semi-homomorphisms and generalized proximity relations between Boolean algebras. *Miskolc Math. Notes*, 19(1):171–189, 2018.
- [7] H. de Vries. *Compact spaces and compactifications. An algebraic approach*. PhD thesis, University of Amsterdam, 1962.
- [8] A. Jung, M. Kegelman, and M. A. Moshier. Stably compact spaces and closed relations. In *MFPS 2001. Papers from the 17th Conference on the Mathematical Foundations of Programming Semantics, Aarhus University, Aarhus, Denmark, May 23–26, 2001*, pages 209–231. Amsterdam: Elsevier, 2001.
- [9] A. Kurz, A. Moshier, and A. Jung. Stone duality for relations. In A. Palmigiano and M. Sadrzadeh, editors, *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, pages 159–215. Springer International Publishing, Cham, 2023.
- [10] M. A. Moshier. On the relationship between compact regularity and Gentzen’s cut rule. *Theor. Comput. Sci.*, 316(1-3):113–136, 2004.
- [11] C. F. Townsend. *Preframe techniques in constructive locale theory*. PhD thesis, Imperial College London (University of London), 1996.
- [12] Y. Venema and J. Vosmaer. Modal logic and the Vietoris functor. In G. Bezhanishvili, editor, *Leo Esakia on duality in modal and intuitionistic logics*, volume 4 of *Outst. Contrib. Log.*, pages 119–153. Springer, Dordrecht, 2014.