

A categorical duality for metrically complete lattice-ordered Abelian groups with unit

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Abstract

This extended abstract presents joint work with Vincenzo Marra and Luca Spada on a Stone–Gelfand duality for metrically complete unital lattice-ordered Abelian groups. We identify a category of compact Hausdorff spaces enriched with local arithmetic data and prove that it is dually equivalent to the algebraic category. The resulting duality simultaneously extends classical Stone duality and Yosida duality, which arise as its two extremal cases.

1 From Stone duality to Yosida duality

Dualities between algebraic and topological structures have played a central role in logic since Stone’s representation theorem for Boolean algebras. Stone duality identifies Boolean algebras with algebras of clopen subsets of Stone spaces and, categorically, yields a contravariant equivalence between Boolean algebras and compact zero-dimensional Hausdorff spaces.

A rather different family of dualities arose in functional analysis. Theorems of Gelfand, Kakutani, Stone, and Yosida represent various ordered or normed algebraic structures as algebras of continuous real- or complex-valued functions on compact Hausdorff spaces. For the present discussion, the most relevant result is Yosida duality: compact Hausdorff spaces correspond contravariantly to metrically complete vector lattices with a distinguished unit.

Stone and Yosida dualities appear to sit at opposite ends of a spectrum. Stone duality concerns highly disconnected spaces and algebraic structures of a discrete character, whereas Yosida duality involves arbitrary compact Hausdorff spaces and real-valued continuous functions. Nevertheless, these two worlds coexist naturally in many-valued logic. By Mundici’s equivalence, MV-algebras—the algebraic semantics of Łukasiewicz logic—correspond to unital lattice-ordered Abelian groups. These groups interpolate between the Boolean and vector-lattice cases, suggesting that Stone and Yosida duality should be extremal instances of a single more general picture.

The aim of our work [1] is to construct such a picture for metrically complete lattice-ordered Abelian groups with unit, abbreviated below as metrically complete ℓ -groups.

2 The representation problem

A unital Abelian ℓ -group is an Abelian group G , equipped with a translation-invariant lattice order and a distinguished positive unit 1. The unit determines a natural pseudometric,

$$d(v, w) = \inf \left\{ \frac{p}{q} \in \mathbb{Q}_{\geq 0} : -p1 \leq q(v - w) \leq p1 \right\},$$

and we consider those groups for which this is a complete metric. The standard example is $C(X, \mathbb{R})$, with the uniform metric, for a compact Hausdorff space X .

Goodearl and Handelman proved that every metrically complete ℓ -group can be represented as a group of continuous functions on a compact Hausdorff space, subject to pointwise restrictions on their values. More precisely, one obtains a compact Hausdorff space X and, for each $x \in X$, a subgroup

$$D_x \in \{\mathbb{R}\} \cup \left\{ \frac{1}{n}\mathbb{Z} : n \geq 1 \right\}$$

such that

$$G \cong C_D(X) := \{f \in C(X, \mathbb{R}) : f(x) \in D_x \text{ for every } x \in X\}.$$

The set D_x records the local arithmetic constraint imposed at x .

This theorem strongly suggests a common generalization of Stone and Yosida duality, but it is not itself a duality. There are two obstructions. First, arbitrary spaces with pointwise restrictions are not uniquely determined by the represented ℓ -group: non-isomorphic enriched spaces may yield isomorphic groups. Second, the theorem does not identify the topological maps corresponding to unit-preserving ℓ -homomorphisms. Thus the problem is to isolate both the correct spatial objects and the correct morphisms.

3 Normal arithmetic spaces

We solve the first problem by introducing *normal a-spaces*, (short for *normal arithmetic spaces*). Such an object consists of a compact Hausdorff space X together with denominator data $x \mapsto D_x$ as above, satisfying two additional requirements.

The first is a separation axiom. Whenever $x \neq y$, there exist disjoint open neighbourhoods U of x and V of y such that every point outside $U \cup V$ is unconstrained, that is, $D_z = \mathbb{R}$ for all $z \notin U \cup V$.

The second is a semicontinuity condition: for every $n \geq 1$, the subset

$$\{x \in X : D_x \subseteq \frac{1}{n}\mathbb{Z}\}$$

is closed. Informally, arithmetic constraints may persist at limit points but cannot become abruptly stronger there.

These axioms are precisely what is needed to make the representation canonical. They also clarify how the new spaces interpolate between the classical cases. If $D_x = \mathbb{R}$ for every x , the additional structure disappears and one recovers an arbitrary compact Hausdorff space. If $D_x = \mathbb{Z}$ for every x , the separation axiom forces clopen subsets to separate points, so that X is a Stone space.

A guiding example is the unit interval with its usual topology and its natural denominator structure:

$$D_x = \mathbb{R} \quad \text{if } x \text{ is irrational,} \quad D_{m/n} = \frac{1}{n}\mathbb{Z}$$

whenever m/n is in lowest terms. This example explains the terminology and displays, within one compact space, the coexistence of continuous and discrete arithmetic behaviour.

A key technical result is an arithmetic refinement of Urysohn's lemma: continuous functions separating suitable closed sets can be chosen so as to respect all local denominator constraints. From this we obtain an analogue of the classical embedding theorem for compact Hausdorff spaces: normal a-spaces are exactly the closed subspaces of powers of the arithmetically enriched unit interval. This result is the main topological ingredient in the duality.

4 The duality

To describe the morphisms, let (X, D) and (Y, E) be normal a-spaces. A continuous map $f : Y \rightarrow X$ is called denominator-decreasing when

$$D_{f(y)} \subseteq E_y \quad \text{for every } y \in Y.$$

This is exactly the condition ensuring that composition with f sends denominator-respecting functions on X to denominator-respecting functions on Y :

$$C_D(X) \longrightarrow C_E(Y), \quad g \longmapsto g \circ f.$$

Our main theorem states that this construction yields a dual equivalence:

The category of metrically complete Abelian ℓ -groups with unit and unit-preserving ℓ -homomorphisms is dually equivalent to the category of normal a-spaces and denominator-decreasing continuous maps.

In particular, every metrically complete ℓ -group has a canonical representation as $C_D(X)$ for a normal a-space (X, D) ; the space is determined uniquely up to isomorphism, and every unit-preserving ℓ -homomorphism is induced uniquely by a denominator-decreasing continuous map in the opposite direction.

The theorem simultaneously recovers two classical dualities. When all denominators are $\mathbb{R}$, the algebraic objects are metrically complete vector lattices and the duality reduces to Yosida duality. When all denominators are $\mathbb{Z}$, the spatial objects are Stone spaces. On the algebraic side one obtains groups of the form $C(X, \mathbb{Z})$; applying Mundici's equivalence and restricting to the unit interval $[0, 1]$ recovers the Boolean algebra of clopen subsets of X , and hence classical Stone duality.

Thus the discrete and continuous representation theories are not separate phenomena. They are the two boundary cases of a single categorical duality whose intermediate objects combine topology with local arithmetic restrictions. Besides providing the first duality for metrically complete ℓ -groups, this framework gives a geometric interpretation of metrically complete MV-algebras and clarifies the structure hidden in the Goodearl–Handelman representation theorem.

References

- [1] M. Abbadini, V. Marra, and L. Spada. Stone–Gelfand duality for metrically complete lattice-ordered groups. *Advances in Mathematics*, 461:110067, 2025.