The opposite of the category of compact ordered spaces is monadic over the category of sets*

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1 Introduction

Dualities have been extensively studied in the last century, one of the aim being to understand algebraic structures through topological ones. In 1936, M.H. Stone described a duality between the category of Boolean algebras and homomorphisms and the category $\mathcal{BooSp}$ of totally disconnected compact Hausdorff spaces and continuous maps, now known as Boolean (or Stone) spaces. Two years later, Stone published a generalization to distributive lattices [11], where the dual category consists of the nowadays called spectral spaces and perfect maps. H.A. Priestley showed in 1970 that spectral spaces can be equivalently described as the nowadays called Priestley spaces, i.e. compact spaces equipped with a partial order satisfying a condition called total order-disconnectedness [9]. Precisely, Priestley duality states that the category of bounded distributive lattices and homomorphisms is dually equivalent to the category $\mathcal{Pries}$ of Priestley spaces and order-preserving continuous maps.

While Boolean spaces and Priestley spaces were introduced because they describe the duals of algebras of prime interest in logic, outside the disconnectedness setting we find structures of independent interest: compact Hausdorff spaces (which generalize Boolean spaces), and so-called compact ordered spaces (which generalize Priestley spaces). A *compact ordered space* (also known as *compact pospace*, or *Nachbin space*) is a compact space X equipped with a partial order that is closed in the product topology of $X \times X$. These structures were first introduced in 1948 by L. Nachbin (about twenty years before Priestley spaces), and are a partially ordered version of compact Hausdorff spaces, just like Priestley spaces are a partially ordered version of Boolean spaces. A standard reference is [8, Chapter I].

The classes of Boolean algebras and of bounded distributive lattices are varieties of finitary algebras, i.e. classes of algebras with operations of finite arity defined by universally quantified equations. Thus, both the opposite of $\mathcal{BooSp}$ and the opposite of $\mathcal{Pries}$ admit a forgetful functor to $\mathcal{Set}$ which is monadic and finitary. One may wonder whether something similar is true for the larger categories $\mathcal{CompHaus}$ of compact Hausdorff spaces and continuous maps and $\mathcal{CompOrd}$ of compact ordered spaces and order-preserving continuous maps.

It is known that $\mathcal{CompHaus}$ is not dually equivalent to any variety of finitary algebras. More is true: M. Lieberman, J. Rosický, and S. Vasey proved that no faithful functor $\mathcal{CompHaus}^{\mathrm{op}} \to \mathcal{Set}$ preserves directed colimits [7], whence $\mathcal{CompHaus}$ is not dually equivalent to any first-order definable class of structures. Therefore, in the passage from Boolean spaces to compact Hausdorff spaces, we lose the finitary nature of the opposite category. However, the algebraic nature is maintained: $\mathcal{CompHaus}$ is monadic over $\mathcal{Set}$. This fact was first mentioned by J. Duskin in 1969 [4, 5.15.3, p. 118] (see [3, Chapter 9, Theorem 1.11, p. 256] for a proof). It is not difficult to show that, like $\mathcal{CompHaus}$, the category $\mathcal{CompOrd}$ of compact ordered spaces and order-preserving continuous maps is not dually equivalent to any variety of finitary algebras. More generally, no faithful functor $\mathcal{CompOrd}^{\mathrm{op}} \to \mathcal{Set}$ preserves directed colimits. The question then arises:

$$\text{Is } \mathcal{CompOrd}^{\mathrm{op}} \text{ monadic over } \mathcal{Set}?$$

This question appeared as open in [6].

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2 Main results

We prove the following key result.

Proposition 1 *Equivalence relations in* $\mathsf{CompOrd}^{\mathrm{op}}$ *are effective.*

Thus, $\mathsf{CompOrd}^{\mathrm{op}}$ is Barr-exact (regularity was already known). While the proof of an analogous statement for compact Hausdorff spaces is quite short (see [3, Chapter 9, Theorem 1.11, p. 256]), it seems that proving Proposition 1 requires considerably more effort: to our understanding, the difference is related to the fact that $\mathsf{CompHaus}^{\mathrm{op}}$ is a Mal'cev category, whereas $\mathsf{CompOrd}^{\mathrm{op}}$ is not. Adding Proposition 1 to other known results, we obtain a positive answer to the open question in [6] mentioned at the end of the introduction.

Theorem 2 *The opposite of the category* $\mathsf{CompOrd}$ *of compact ordered spaces and order-preserving continuous maps is monadic over* Set*.*

Theorem 2 can be used to prove the following statement, which strengthens a result in [6].

Theorem 3 *The opposite of the category of coalgebras for the Vietoris endofunctor on* $\mathsf{CompOrd}$ *is monadic over* Set*.*

Using Proposition 1, we obtain a bijection between the set of equivalence relations on a compact ordered space X seen as an object of $\mathsf{CompOrd}^{\mathrm{op}}$ and the set of closed subspaces of X. As mentioned above, $\mathsf{CompOrd}^{\mathrm{op}}$ is not Mal'cev; we characterize the commuting equivalence relations in $\mathsf{CompOrd}^{\mathrm{op}}$, generalizing an analogous fact for Priestley spaces and bounded distributive lattices in [5, Lemma 5.4].

Proposition 4 *Let* C_1 *and* C_2 *be closed subspaces of a compact ordered space* X*. The corresponding equivalence relations on the object* X *seen as an object of* $\mathsf{CompOrd}^{\mathrm{op}}$ *commute iff, for every* $x_1 \in C_1$, $x_2 \in C_2$, $\{i,j\} = \{1,2\}$ *with* $x_i \leq x_j$*, there exists* $z \in C_1 \cap C_2$ *such that* $x_i \leq z \leq x_j$*.*

Propositions 1 and 2 can be found in [2], and Theorem 3 can be found in [1].

References

[1] M. Abbadini. On the axiomatisability of the dual of compact ordered spaces. PhD Thesis, University of Milan

[2] M. Abbadini and L. Reggio. On the Axiomatisability of the Dual of Compact Ordered Spaces. *Applied Categorical Structures* 28(2020)921—934

[3] M. Barr, and C. Wells. Toposes, triples and theories. Springer-Verlag New York. Republished in: *Repr. Theory Appl. Categ.* 12(2005)1–288

[4] J. Duskin. Variations on Beck's tripleability criterion. In S. Mac Lane, editor, Reports of the Midwest Category Seminar, III, pages 74–129. Springer, Berlin

[5] M. Gehrke, S.J. v. Gool. Sheaves and duality. *J. Pure Appl. Algebra* 222(2018)2164–2180

[6] D. Hofmann, R. Neves, P. Nora. Generating the algebraic theory of $C(X)$: the case of partially ordered compact spaces. *Theory and Applications of Categories* 33(2018)276–295

[7] M. Lieberman, J. Rosický, S. Vasey. Hilbert spaces and C^*-algebras are not finitely concrete. *Preprint available at arXiv:1908.10200*

[8] L. Nachbin. Topology and order. Translated from the Portuguese by Lulu Bechtolsheim. Van Nostrand Mathematical Studies, No. 4. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London

[9] H.A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.* 2(1970)186–190

[10] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.* 40(1936)37–111

[11] M. H. Stone. Topological representations of distributive lattices and brouwerian logics. *Časopis pro pěsiování matematiky a fysiky* 67(1938)1–25