The opposite of the category of compact ordered spaces as an infinitary variety

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1 Background and main question

In 1936, M. H. Stone described what is nowadays known as Stone duality for Boolean algebras [12]. In modern terms, the result states that the category of Boolean algebras and homomorphisms is dually equivalent to the category of totally disconnected compact Hausdorff spaces and continuous maps, now known as Stone or Boolean spaces.

If we drop the assumption of total disconnectedness, we are left with the category CH of compact Hausdorff spaces and continuous maps. J. Duskin observed in 1969 that the opposite category $\mathsf{CH}^{\mathsf{op}}$ is monadic over the category Set of sets and functions [5, 5.15.3]. In fact, $\mathsf{CH}^{\mathsf{op}}$ is equivalent to a variety of algebras with primitive operations of at most countable arity: a finite generating set of operations was exhibited by J. Isbell [7], while a finite equational axiomatisation was provided by V. Marra and L. Reggio [9]. Therefore, if we allow for infinitary operations, Stone duality can be lifted to compact Hausdorff spaces, retaining the algebraic nature of the category involved.

In 1970, H. A. Priestley introduced what are now known as Priestley spaces, i.e. compact spaces equipped with a partial order satisfying a condition called total order-disconnectedness, and showed that the category of bounded distributive lattices and homomorphisms is dually equivalent to the category of Priestley spaces and order-preserving continuous maps [11].

The category of Priestley spaces is a full subcategory of the category $\mathsf{CompOrd}$ of compact ordered spaces and order-preserving continuous maps: here, by a *compact ordered space*, we mean a compact space equipped with a partial order which is closed in the product topology—a partially ordered version of compact Hausdorff spaces introduced by L. Nachbin in 1948 [10].

Similarly to the case of Boolean algebras, one may ask if Priestley duality can be lifted to $\mathsf{CompOrd}$ retaining the algebraic nature of the opposite category. In other words:

> Is the category $\mathsf{CompOrd}$ of compact ordered spaces dually equivalent to a variety of (possibly infinitary) algebras?

This appeared as an open question in a paper by D. Hofmann, R. Neves and P. Nora [6].

2 Results

The following is our main result.

Theorem 1. *The category* $\mathsf{CompOrd}$ *of compact ordered spaces and order-preserving continuous maps is dually equivalent to a variety of algebras, with operations of at most countable arity.*

This gives a positive answer to the open question in [6]. This result was first proved by the author in [1], but a shorter proof was obtained in a joint work with L. Reggio [4]. A natural way to describe a variety dual to $\mathsf{CompOrd}$ uses the signature Σ of all the order-preserving continuous maps from finite or countably infinite powers of $[0,1]$ to $[0,1]$ itself: it was already known that $\mathsf{CompOrd}^{\mathsf{op}}$ is equivalent to the class $\mathbb{SP}([0,1])$ of subalgebras of powers of $[0,1]$ (with obvious interpretation of the function symbols in Σ), and via categorical means we prove that this class is closed under homomorphic images and thus it is a variety of (infinitary) algebras.

The countable bound on the arity is the best possible, since CompOrd is not dually equivalent to any variety of finitary algebras. Indeed, the following stronger results hold:

1. CompOrd is not dually equivalent to any finitely accessible category;
2. CompOrd is not dually equivalent to any first-order definable class (as suggested by S. Vasey as an application of a result by M. Lieberman, J. Rosický and S. Vasey [8]);
3. CompOrd is not dually equivalent to any class of finitary algebras closed under products and subalgebras.

Manageable sets of primitive operations and equational axioms for $\mathsf{CompOrd}^{\mathsf{op}}$ exist: we exhibit a *finite* equational axiomatisation of $\mathsf{CompOrd}^{\mathsf{op}}$, meaning that we use only finitely many function symbols (of at most countable arity) and finitely many equational axioms to present the variety.

To conclude, we recall that D. Hofmann, R. Neves and P. Nora proved that the opposite of the category of coalgebras for the Vietoris endofunctor on CompOrd is equivalent to an $\aleph_1$-ary quasivariety [6]. Our results can be used to show that this is actually a variety.

The results described in this abstract are part of the author's doctoral thesis [3], supervised by V. Marra at the University of Milan. Some of them are covered in [1, 2, 4].

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