PRIESTLEY DUALITY ABOVE DIMENSION ZERO: ALGEBRAIC AXIOMATISABILITY OF THE DUAL OF COMPACT ORDERED SPACES

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Stone showed that the category of Boolean algebras with homomorphisms is dually equivalent to the category of totally disconnected compact Hausdorff spaces with continuous maps [6]. If we drop the assumption of total disconnectedness, we are left with the category KH of compact Hausdorff spaces and continuous maps. Duskin showed that the opposite category $\mathsf{KH}^{\mathrm{op}}$ is monadic over the category of sets and functions [1, 5.15.3]. In fact, $\mathsf{KH}^{\mathrm{op}}$ is equivalent to a variety of algebras with primitive operations of countable arity; [3] exhibits a generating set of operations, while [4] provides a finite axiomatisation. Thus, Stone duality can be lifted to compact Hausdorff spaces, retaining the algebraic nature.

Stone established also a duality for (bounded) distributive lattices and homomorphism with the so-called spectral spaces and perfect maps [7]. While spectral spaces are non-Hausdorff, Priestley showed that they can be equivalently described as certain partially ordered topological spaces [5]. More precisely, the category of distributive lattices is dually equivalent to the full subcategory of Nachbin's compact ordered spaces on the totally order-disconnected objects. A *compact ordered space* is a pair $(X, \leq)$ where X is a compact space, and $\leq$ is a partial order on X, closed in the product topology of $X \times X$; a morphism is a monotone continuous map. Similarly to the case of Boolean algebras, one may ask if Priestley duality can be lifted to the category $\mathsf{KH}_{\leq}$ of compact ordered spaces, retaining its algebraic nature. In [2], $\mathsf{KH}_{\leq}^{\mathrm{op}}$ is shown to be equivalent to a quasi-variety of (infinitary) algebras, leaving as open the question whether it is equivalent to a variety. We show that the answer is affirmative.

Theorem 1 (Main result)**.** *The opposite of the category of compact ordered spaces is equivalent to a variety of (infinitary) algebras.*

To prove the theorem, we use the fact that a quasi-variety is a variety if, and only if, every equivalence relation is effective. First, we characterise equivalence relations on a compact ordered space X in the category $\mathsf{KH}_{\leq}^{\mathrm{op}}$ as certain pre-orders on the

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order-topological coproduct $X + X$. Then, we rephrase effectiveness into an order-theoretic condition, and show that it is satisfied by every pre-order arising from an equivalence relation.

We also show that it is necessary to resort to infinitary operations.

Theorem 2. *The opposite of the category of compact ordered spaces is not equivalent to a variety of finitary algebras.*

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