Norm complete Abelian ℓ-groups: equational axiomatization

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1 Introduction

An *Abelian lattice-ordered group* (or *ℓ-group*, for short) is an Abelian group G, endowed with a partial lattice order $\leq$ that is translation invariant, i.e., for all $x, y, z \in G$, if $x \leq y$, then $x + z \leq y + z$. An element u of an ℓ-group is a (strong order) unit if, for all $x \in G$, there exists $n \in \mathbb{N}$ such that $|x| \leq nu$. A *unital ℓ-group* is an ℓ-group G with a designated unit u, and a *morphism of unital ℓ-groups* is a map that preserves the lattice structure, the group structure, and the unit.

An undesired issue about unital ℓ-groups is that, in their usual presentation, they fail to be an elementary class. The problem is essentially due to the fact that the definition of unit may not be expressed in first-order logic, as an application of the compactness theorem shows. However D. Mundici showed in [4] that the category of unital ℓ-groups is equivalent to a finitary variety finitely axiomatized: the category of MV-algebras. In particular, an MV-algebra $\langle A, \oplus, \neg, 0\rangle$ is a set A, equipped with a binary operation $\oplus$, a unary operation $\neg$ and a distinguished constant 0 such that $\langle A, \oplus, 0\rangle$ is a commutative monoid, $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. We have a funtor Γ —which is proved to be an equivalence in [4]— from the category of unital ℓ-groups to the category of MV-algebras: for (G, u) a unital ℓ-group, $\Gamma((G, u)) \coloneqq \{x \in G \mid 0 \leq x \leq u\}$, where, for $x, y \in \Gamma((G, u))$, $x \oplus y \coloneqq (x + y) \wedge u$, and $\neg x \coloneqq u - x$.

Every unital ℓ-group (G, u) carries a natural seminorm $\|x\| \coloneqq \inf\left\{\frac{p}{q} \in \mathbb{Q}^+ \mid qx \leq pu\right\}$, which induces a pseudometric $\mathrm{d}(x, y) \coloneqq \|x - y\|$. What is missing for d to be a metric is the implication $\mathrm{d}(x, y) = 0 \Rightarrow x = y$. This happens precisely when G is Archimedean, i.e. when, for all $x, y \in G$, if, for all $n \in \mathbb{N}$, $n|x| \leq y$, then $x = 0$. We write *norm complete ℓ-group* for "Archimedean unital ℓ-group complete in the metric d". A *morphism of norm complete ℓ-groups* is simply a morphism of unital ℓ-groups.

2 Main results

Our main result is the following.

Theorem 2.1. *Up to an equivalence, the category of norm complete ℓ-groups is an (infinitary) variety of algebras.*

Theorem 2.1 is analogous to the well-known result that, up to an equivalence of categories, the category of norm complete vector lattices is an (infinitary) variety of algebras (see [1], [2] and [3]). The difference here is —roughly speaking— that divisibility is not required.

We obtain Theorem 2.1 in two steps.

2.1 First step

As a first step, we notice —as stated in Proposition 2.2 below— that the category of norm complete ℓ-groups is equivalent to the category of norm complete MV-algebras, whose definition we make more precise now. On any MV-algebra we can define a pseudometric d that coincides with the restriction of the pseudometric d on (G, u), where (G, u) is a unital ℓ-group such that $\Gamma((G, u)) \cong A$; the pseudometric d on A is a metric if, and only if, (G, u) is Archimedean if, and only if, A is an *Archimedean MV-algebra*, i.e., for any $x \in A$, if, for all $n \in \mathbb{N}$, $\underbrace{x \oplus \cdots \oplus x}_{n \text{ times}} \leq \neg x$, then $x = 0$. We write *norm complete MV-algebra* for "Archimedean MV-algebra complete in the metric d". A *morphism of norm complete MV-algebras* is simply a morphism of MV-algebras.

Proposition 2.2. *The functor Γ from unital ℓ-groups to MV-algebras restricts to an equivalence between norm complete ℓ-groups and norm complete MV-algebras.*

2.2 Second step

As a second step, we provide an equational axiomatization defining a variety CMV which is isomorphic to the category of norm complete MV-algebras. This variety is not finitary: the set of primitive operations that we consider is made of a set of primitive operations of MV-algebras, together with an operation γ of countably infinite arity. The idea is that, in the intended models, $\gamma(x_0, x_1, x_2, \dots) = \lim_{n\to\infty} a_n$ whenever $(x_n)_n$ converges "quickly enough"; precisely, when, for all $n \in \mathbb{N}$, $\mathrm{d}(x_n, x_{n+1}) \leq \frac{1}{2^{n+1}}$. We discuss a proof of the following

Theorem 2.3. *The category of norm complete MV-algebras is isomorphic to the variety* CMV.

2.3 Conclusion

When coupled with the content of Luca Spada's talk "Norm complete Abelian ℓ-groups: topological duality", the result presented here amounts to the following duality theorem, that extends Stone-Gelfand duality to a-normal spaces.

Theorem 2.4. *The following categories are pairwise equivalent.*

1. *The opposite of the category of a-normal spaces with a-maps among them.*
2. *The category of norm complete ℓ-groups.*
3. *The category of norm complete MV-algebras.*
4. *The (infinitary) variety* CMV.

References

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