# OPERATIONS THAT PRESERVE INTEGRABILITY, AND TRUNCATED RIESZ SPACES

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For $(\Omega, \mathcal{F}, \mu)$ a measure space, we set $\mathcal{L}^1(\mu) \coloneqq \{f\colon \Omega \to \mathbb{R} \mid f \text{ is } \mathcal{F}\text{-measurable and } \int_\Omega |f| d\mu < \infty\}$. The aim of this contribution is to determine and axiomatize the equational algebraic structure of such $\mathcal{L}^1$ spaces. It is well known that $\mathcal{L}^1(\mu)$ is closed under pointwise sum but, in general, it is not closed under pointwise product. Given a function $\tau\colon \mathbb{R}^I \to \mathbb{R}$, we show that $\mathcal{L}^1(\mu)$ is closed under $\tau$ for every measure $\mu$ iff, roughly speaking, $\tau$ is measurable and at most linear. In such case, we say that $\tau$ *preserves integrability.*

Let $\mathcal{V}$ be the infinitary variety whose operations are the functions $\mathbb{R}^I \to \mathbb{R}$ (with $I \in \mathbb{N} \cup \{\mathbb{N}\}$) that preserve integrability, and whose axioms are the equations satisfied by $\mathbb{R}$. Main examples of objects in $\mathcal{V}$ are the $\mathcal{L}^1$ spaces. We show that $\mathcal{V}$ is isomorphic to the category of Dedekind $\sigma$-complete truncated Riesz spaces, where "truncated" is intended in the sense of R. Ball [1]. As primitive operations one can take those of Riesz spaces, Ball's truncation, along with a truncated version of countable joins. A simple axiomatization of this variety is given. The free object on a set $I$ in $\mathcal{V}$ is shown to be $Free_I \coloneqq \{\tau\colon \mathbb{R}^I \to \mathbb{R} \mid \tau \text{ preserves integrability}\}$.

Analogous results are obtained for the spaces $\mathcal{L}^1(\mu)$, with $\mu$ finite. In particular, we show that the corresponding infinitary variety is isomorphic to the category of Dedekind $\sigma$-complete Riesz spaces with weak unit. (These results are part of ongoing work towards a Ph.D. degree under the supervision of V. Marra.)

## References

[1] Richard N. Ball. Truncated abelian lattice-ordered groups I: The pointed (Yosida) representation. *Topology Appl.*, 162:43–65, 2014.