Stone-Gelfand duality for groups

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1 Introduction

A question, which, to the best of our knowledge, is still open, is the following: is the opposite of the category $\mathsf{CompHaus}$ of compact Hausdorff spaces with continuous maps equivalent to an elementary class? Approximations of a negative answer were given (see, for example, [1], [2], [7]); in particular, $\mathsf{CompHaus}^{\mathrm{op}}$ is not equivalent to a finitary variety of algebras; that is, it is impossible to equationally axiomatize it by operations of finite arity. However, $\mathsf{CompHaus}^{\mathrm{op}}$ is equivalent to an infinitary variety Δ whose primitive operations are of at most countable arity (see [3], [6], [7]).

A related result is the classical Stone-Gelfand duality, which asserts that $\mathsf{CompHaus}^{\mathrm{op}}$ is equivalent to the category $\mathsf{ComplVectLatt}$ of archimedean vector lattices, equipped with a (strong order) unit, and complete in the unit norm, with unit-preserving vector lattice homomorphisms. Therefore, we have a chain of equivalences:

$$\mathsf{CompHaus}^{\mathrm{op}} \cong \mathsf{ComplVectLatt} \cong \Delta. \tag{1}$$

In this work, we replace the linear structure of the objects of $\mathsf{ComplVectLatt}$ with a weaker one – namely, the structure of abelian groups. We write "norm complete ℓ-group" for "archimedean lattice-ordered abelian group equipped with a (strong order) unit, and complete in the unit norm". We denote by $\mathsf{Compl}\ell\mathsf{Gr}$ the category of norm complete ℓ-groups, with unit preserving group lattice homomorphism. We address the question: is there, analogously to (1), a chain of equivalences for $\mathsf{Compl}\ell\mathsf{Gr}$ of the form

$$\mathsf{K}^{\mathrm{op}} \overset{?}{\cong} \mathsf{Compl}\ell\mathsf{Gr} \overset{?}{\cong} \mathcal{V}. \tag{2}$$

for some category K "of spaces" and some (possibly infinitary) variety $\mathcal{V}$?

2 Main results

2.1 Equational axiomatization for norm complete ℓ-groups

Theorem 2.1. *The category* $\mathsf{Compl}\ell\mathsf{Gr}$ *of norm complete ℓ-groups is equivalent to an (infinitary) variety.*

We call CMV the variety in Theorem 2.1 (for "complete MV-algebras"). The set of primitive operations of CMV that we consider is made of a set of primitive operations of MV-algebras, together with an operation γ of countably infinite arity. The idea is that, in the intended models, $\gamma(x_0, x_1, x_2, \dots) = \lim_{n\to\infty} a_n$ whenever $(x_n)_n$ converges "quickly enough"; precisely, when, for all $n \in \mathbb{N}$, $\mathrm{dist}(x_n, x_{n+1}) \leq \frac{1}{2^{n+1}}$.

2.2 Topological duality for norm complete ℓ-groups

To establish a duality that relates norm complete ℓ-groups to a certain category of "spaces", a first important result was proved by Goodearl and Handelman [5, Theorem 5.5]:

Theorem 2.2. *Let X be a compact Hausdorff space and $C(X,\mathbb{R})$ be the set of continuous functions from X into $\mathbb{R}$. For each $x \in X$, let A_x be either $\mathbb{R}$ or $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N} \setminus \{0\}$. Set*

$$D = \{f \in C(X,\mathbb{R}) \mid f(x) \in A_x \text{ for all } x \in X\},$$

and give to D the ℓ-group structure inherited from $C(X,\mathbb{R})$. Then D is a norm complete ℓ-group. Conversely, any norm complete ℓ-group is isomorphic to one of this form.

The crucial restriction to functions such that $f(x) \in A_x$ can be understood as a *labelling* on the space X that must be respected by the continuous functions considered. The set of labels is $\{\mathbb{R}, \frac{1}{1}\mathbb{Z}, \frac{1}{2}\mathbb{Z}, \frac{1}{3}\mathbb{Z}, \dots\}$; for convenience, we identify this set with $\mathbb{N} = \{0,1,2,3,\dots\}$, associating to $\mathbb{R}$ the number 0, and to $\frac{1}{n}\mathbb{Z}$ the number n. For $q \in \mathbb{R}$, we write $\mathtt{den}(q)$ to denote the denominator of q (in its irreducible form) if $q \in \mathbb{Q}$, and 0 otherwise. In this way, $\frac{1}{n}\mathbb{Z} = \{q \in \mathbb{R} \mid \mathtt{den}(q) \text{ divides } n\}$, and $\mathbb{R} = \{q \in \mathbb{R} \mid \mathtt{den}(q) \text{ divides } 0\}$. Hence, Theorem 2.2 may be restated as follows.

Theorem 2.3. *Let X be a compact Hausdorff space and $C(X,\mathbb{R})$ be the set of continuous functions from X into $\mathbb{R}$. Let $\zeta \colon X \to \mathbb{N}$ be a function. Set*

$$D = \{f \in C(X,\mathbb{R}) \mid \mathtt{den}(f(x)) \text{ divides } \zeta(x)\},$$

and give to D the ℓ-group structure inherited from $C(X,\mathbb{R})$. Then D is a norm complete ℓ-group. Conversely, any norm complete ℓ-group is isomorphic to one of this form.

This serves as motivation for the definition below. For $n \in \mathbb{N}$ we shall write $\mathtt{div}(n)$ for the set of natural numbers that divide n.

Definition 2.4. An *a-normal space* is a compact Hausdorff space X, endowed with a function $\zeta \colon X \to \mathbb{N}$, such that the following conditions hold.

1. For every $n \in \mathbb{N}$, $\zeta^{-1}[\mathtt{div}(n)]$ is closed in the topology τ.
2. For every disjoint closed subsets A and B of X, there exist two open disjoint neighbourhoods U and V of A and B, respectively, such that for every $x \in X \setminus (U \cup V)$, $\zeta(x) = 0$.

A function $f \colon (X,\zeta) \to (X',\zeta')$ between a-normal spaces *respects the denominators* if, for every $x \in X$, $\zeta'(f(x))$ divides $\zeta(x)$. We denote by ANorm the category of a-normal spaces with continuous denominator-respecting maps among them.

Theorem 2.5. *The category $\mathsf{Compl\ell Gr}$ of norm complete ℓ-groups is dually equivalent to the category ANorm of a-normal spaces.*

In conclusion:

$$\mathsf{ANorm}^{\mathrm{op}} \cong \mathsf{Compl\ell Gr} \cong \mathsf{CMV}. \tag{3}$$

References

[1] B. Banaschewski. More on compact Hausdorff spaces and finitary duality. *Canad. J. Math.*, 36(6):1113–1118, 1984.

[2] P. Bankston. Some obstacles to duality in topological algebra. *Canad. J. Math.*, 34(1):80–90, 1982.

[3] J. Duskin. Variations on Beck's tripleability criterion. In *Reports of the Midwest Category Seminar, III*, pages 74–129. Springer, Berlin, 1969.

[4] K. R. Goodearl. Partially ordered abelian groups with interpolation. Mathematical Surveys and Monographs, 20, American Mathematical Society, Providence, RI, 1986.

[5] K. R. Goodearl and D. E. Handelman. Metric completions of partially ordered abelian groups. *Indiana Univ. Math. J.*, 29(6):861–895, 1980.

[6] J. Isbell. Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155, 1982.

[7] V. Marra and L. Reggio. Stone duality above dimension zero: Axiomatizing the algebraic theory of $C(X)$. Advances in Mathematics, 307:253-287, 2017.