# An approach à la de Vries for compact Hausdorff spaces and closed relations

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*TACL 2022, Coimbra, Portugal*
*June 24, 2022*

Stone duality for Boolean algebras [Stone, 1936] states that the category of Boolean algebras is dually equivalent to the category of Stone spaces (= compact Hausdorff spaces with a basis of clopens).

De Vries obtained a duality (nowadays called de Vries duality) for the category **KHaus** of compact Hausdorff spaces and continuous functions [de Vries, 1962].

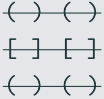
A regular open subset of a space $X$ is a subset $U$ of $X$ such that $U = \text{int}(\text{cl}(U))$ (in particular, it is open).

**Example of regular open subset**

$U$

$\text{cl}(U)$

$\text{int}(\text{cl}(U)) = U$

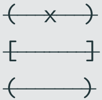


**Example of non-(regular open) subset**

$U$

$\text{cl}(U)$

$\text{int}(\text{cl}(U))$



For any space $X$, the set $\mathrm{RO}(X)$ of regular open subsets of $X$ is a complete boolean algebra with respect to the inclusion order [MacNeille, 1937], [Tarski, 1937].

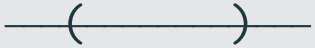
$$\begin{aligned} A \vee B &= \mathrm{int}(\mathrm{cl}(A \cup B)); \\ A \wedge B &= A \cap B; \\ 0 &= \emptyset; \\ 1 &= X; \\ \neg A &= \mathrm{int}(X \setminus A). \end{aligned}$$

To a compact Hausdorff space $X$, de Vries associated the boolean algebra $\mathrm{RO}(X)$, equipped with the <u>well-inside</u> relation $\prec$:

$$A \prec B \iff \mathrm{cl}(A) \subseteq B.$$

**Example**

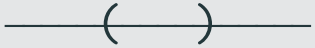
$U :=$ 

$V :=$ 

$U \prec V$.

**Example**

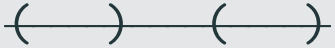
$W :=$ 

$W :=$ 

$W \not\prec W$.

**Example**

$A :=$ 

$B :=$ 

$A \not\prec B$.

$X$ can be recast from the structure of boolean algebra of $\mathrm{RO}(X)$ together with the relation $\prec$.

**Definition ([de Vries, 1962])**

A de Vries algebra is a complete boolean algebra equipped with a binary relation $\prec$ (called proximity) s.t.:

1. $a \prec 1$;
2. $(a \prec b,\ a \prec c)$ implies $a \prec b \wedge c$;
3. $a \prec b$ implies $\neg b \prec \neg a$;
4. $a \prec b$ implies $a \leq b$;
5. $a \leq b \prec c \leq d$ implies $a \prec d$;
6. $a \prec b$ implies that there exists $c$ such that $a \prec c \prec b$.
7. $a \neq 0$ implies that there exists $b \neq 0$ such that $b \prec a$.

For every compact Hausdorff space $X$, $(\mathrm{RO}(X), \prec)$ is a de Vries algebra. Every de Vries algebra is isomorphic to $(\mathrm{RO}(X), \prec)$ for some compact Hausdorff space $X$ (unique up to homeomorphism) [de Vries, 1962].

To a continuous function $f \colon X \to Y$ between compact Hausdorff spaces, de Vries associates the function

$$\begin{aligned} f^* \colon \mathrm{RO}(Y) &\longrightarrow \mathrm{RO}(X) \\ V &\longmapsto \mathrm{int}(\mathrm{cl}(f^{-1}[V])). \end{aligned}$$

This leads to a duality between **KHaus** and a category **DeV** whose objects are de Vries algebras, and whose morphisms are functions satisfying certain properties [de Vries, 1962]. However,

> Composition of morphisms in **DeV** is <u>not</u> usual function composition.

Our proposal:

We work with certain relations as morphisms between de Vries algebras.

Advantage: composition of morphisms is usual relation composition.

Working with relations instead of functions is not a new thing: see e.g. the (dual) equivalences in

1. [Abramsky, Jung, 1994] for spectral spaces,
2. [Jung, Sünderhauf, 1996] for stably compact spaces,
3. [Moshier, 2004], for compact Hausdorff spaces.

We implement the idea of using relations in the context of de Vries duality.

Given a continuous function $f\colon X \to Y$ between compact Hausdorff spaces, we define a relation $S_f\colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$, as follows:

$$U \; S_f \; V \iff \mathrm{cl}[U] \subseteq f^{-1}[V] \iff f[\mathrm{cl}(U)] \subseteq V.$$

For example, if $f\colon X \to X$ is the identity, then

$$U \; S_f \; V \iff U \prec V.$$

**Definition**

A relation $S\colon A \to B$ between de Vries algebras is called a functional compatible subordination if

1. $S$ is a subordination:
    1.1 $0 \; S \; b$;
    1.2 $a \; S \; 1$;
    1.3 if $a_1 \; S \; b$ and $a_2 \; S \; b$, then $(a_1 \vee a_2) \; S \; b$;
    1.4 if $a \; S \; b_1$ and $a \; S \; b_2$, then $a \; S \; (b_1 \wedge b_2)$;
    1.5 if $a' \leq a \; S \; b \leq b'$, then $a' \; S \; b'$;
2. $S$ is compatible (with the relations $\prec_A$ and $\prec_B$):

$$a \; S \; b \iff \exists \; a' \in A : a \prec_A a' \; S \; b \iff \exists b' \in B : a \; S \; b' \prec_B b;$$

3. $S$ is functional:
    3.1 if $a \; S \; 0$, then $a = 0$;
    3.2 if $b_1 \prec_B b_2$, then there is $a \in A$ s.t. $\neg a \; S \; \neg b_1$ and $a \; S \; b_2$.

Given relations $X \xrightarrow{R} Y \xrightarrow{S} Z$, their composite $S \circ R \colon X \to Z$ is defined by

$$x \,(S \circ R)\, z \iff \exists y \in Y \text{ s.t. } x \, R \, y \, S \, z.$$

**Definition**

We let $\mathbf{DeV^F}$ denote the category

- whose objects are de Vries algebras, and
- whose morphisms are functional compatible subordinations.

Composition is usual composition of relations.

**Main Theorem (1/2)**

The category **KHaus** of compact Hausdorff spaces and continuous functions is equivalent to the category $\mathbf{DeV}^{\mathbf{F}}$ of de Vries algebras and functional compatible subordinations.

Equivalence vs duality: a matter of taste: slightly modifying the functionality axioms one obtains a duality.

Advantage over classical de Vries duality: composition of morphisms is the usual composition of relations.

**Definition**

We let $\mathbf{KHaus^R}$ denote the category

- whose objects are compact Hausdorff spaces, and
- whose morphisms from $X$ to $Y$ are the closed relations $R \subseteq X \times Y$.

Composition of morphisms is the usual composition of relations.

**Main Theorem (2/2)**

The category $\mathbf{KHaus}^{\mathbf{R}}$ of compact Hausdorff spaces and closed relations is equivalent to the category $\mathbf{DeV}^{\mathbf{S}}$ of de Vries algebras and compatible subordinations.

# To sum up

Taking certain relations (instead of functions) as morphisms between de Vries algebras solves some issues.

Furthermore, this approach allows to obtain an equivalence/duality for the category of compact Hausdorff spaces and closed relations between them.

M. Abbadini, G. Bezhanishvili, L. Carai
*A generalization of de Vries duality to closed relations between compact Hausdorff spaces*
Arxiv preprint at `arxiv.org/abs/2206.05711`
(2022)

Thank you.

# Backup slides on piggyback on an equivalence for Stone spaces and closed relations

We piggyback on a generalization of Stone and Halmos duality.

Stone duality = duality for Stone spaces and continuous functions between them.

Halmos duality = duality for Stone spaces and continuous relations between them.

The generalization we need is an equivalence for Stone spaces and closed relations between them (see [Celani, 2018]), that we recall in the next slides.

**Definition**

We let $\textbf{Stone}^{\textbf{R}}$ denote the category of Stone spaces and closed relations between them. Composition is composition of relations. The identity morphism is the equality relation.

To a Stone space $X$ one associates the boolean algebra $\mathrm{Clop}(X)$. To a closed relation $R \colon X \to Y$ one associates the relation $S_R \colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ defined by

$$U \; S_r \; V \iff R[U] \subseteq V.$$

**Definition**

A <u>subordination</u> $S\colon A \to B$ between boolean algebras is a relation s.t.

1. $0 \; S \; b$;
2. $a \; S \; 1$;
3. if $a_1 \; S \; b$ and $a_2 \; S \; b$, then $(a_1 \vee a_2) \; S \; b$;
4. if $a \; S \; b_1$ and $a \; S \; b_2$, then $a \; S \; (b_1 \wedge b_2)$;
5. if $a' \leq a \; S \; b \leq b'$ then $a' \; S \; b'$;

This generalizes the notion of subordination on a boolean algebra in [Bezh., Bezh., Sour., Ven., 2017].

**Definition**

We let $\mathbf{BA^S}$ denote the category of boolean algebras and subordinations between them. Composition of morphisms is relation composition. The identity morphism on an object $A$ is the order $\leq$.

**Theorem**

*The categories* $\mathbf{Stone}^{\mathbf{R}}$ *and* $\mathbf{BA}^{\mathbf{S}}$ *are equivalent (and also dually equivalent).*

Our equivalence between $\mathbf{KHaus}^{\mathbf{R}}$ and $\mathbf{DeV}^{\mathbf{S}}$ is a consequence (and then also the equivalence between $\mathbf{KHaus}$ and $\mathbf{DeV}^{\mathbf{F}}$ follows), as explained in the next slides.

A De Vries algebra can be seen as a pair $(A, \prec)$ where $A$ is a boolean algebra (so, an object of $\mathbf{BA^S}$) and $\prec$ is a subordination from $A$ to $A$ (so, an endomorphism on $A$ in $\mathbf{BA^S}$) satisfying additional conditions; for example, it is idempotent: $\prec \circ \prec = \prec$.

**Definition ([Freyd, 1964])**

The Karoubi envelope (or splitting by idempotents or Cauchy completion) of a category $\mathbf{C}$ is the category $\mathbf{K}(\mathbf{C})$

- whose objects are pairs $(X, f)$, where $X \in \mathbf{C}$ and $f$ is an endomorphism of $X$ such that $f \circ f = f$, and
- whose morphisms from $(X_1, f_1)$ to $(X_2, f_2)$ are the morphisms $g\colon X_1 \to X_2$ in C such that $f_2 \circ g = g = g \circ f_1$.

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ {\scriptstyle f_1}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[g]{} & X_2 \end{array}$$

Composition is composition in $\mathbf{C}$. The identity on $(X, f)$ is $f$.

Every de Vries algebra is an object of $\mathbf{K}(\mathbf{BA^S})$. A morphism $(A, \prec) \to (B, \prec)$ in $\mathbf{K}(\mathbf{BA^S})$ between de Vries algebras is a compatible subordination $S\colon A \to B$.

$$\mathbf{Stone^R} \xleftrightarrow{\text{equiv.}} \mathbf{BA^S}$$

$$\begin{array}{ccccc}
 & & \mathbf{K(Stone^R)} & \xleftrightarrow{\text{equiv.}} & \mathbf{K(BA^S)} \\
 & & \text{full}\big\uparrow & & \text{full}\big\uparrow \\
\mathbf{KHaus^R} & \xleftrightarrow{\text{equiv.}} & \mathbf{Gle^R} & \xleftrightarrow{\text{equiv.}} & \mathbf{DeV^S} \\
\text{wide}\big\uparrow & & \text{wide}\big\uparrow & & \text{wide}\big\uparrow \\
\mathbf{KHaus} & \xleftrightarrow{\text{equiv.}} & \mathbf{Gle} & \xleftrightarrow{\text{equiv.}} & \mathbf{DeV^F}.
\end{array}$$

A Gleason space [Bezh., Bezh., Sour., Ven., 2017] is a pair $(X, E)$ with $X$ a Stone space and $E$ a closed equivalence relation on $X$ s.t. $X \to X/E$ is a Gleason cover of $X/E$. Gleason spaces are objects of $\mathbf{K(Stone^R)}$.

$\mathbf{Gle^R}$ := category of Gleason spaces and “compatible” closed relations [Bezh., Gab., Hard., Jibl., 2019]. $\mathbf{Gle^R}$ is equivalent to $\mathbf{KHaus^R}$ (mapping $(X, E)$ to $X/E$).

A similar usage of Karoubi envelopes in the context of stably compact spaces was mentioned in [Kegelmann, 2002] (and suggested by P. Taylor) and employed in [van Gool, 2012].

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