Duality for metrically complete Abelian ℓ-groups

Joint work with V. Marra and L. Spada

(based on `https://arxiv.org/pdf/2210.15341.pdf`)

Marco Abbadini

Department of Mathematics
University of Salerno

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In the 30s and 40s, a series of papers ([Gelfand and Kolmogorov, 1939], [Krein and Krein, 1940], [Kakutani, 1941], [Stone, 1941], [Yosida, 1941], [Gelfand and Neumark, 1943]) showed that the opposite of the category

$$\mathsf{KHaus} \coloneqq \text{category of compact Hausdorff spaces and continuous maps}$$

can be represented, to within an equivalence, in several useful ways that variously relate to known mathematical structures.

The common idea is to associate with a compact Hausdorff space X the set $C(X, \mathbb{R})$ of continuous real-valued functions (or the set $C(X, \mathbb{C})$ of continuous complex-valued functions).

The mathematical structures used in these representation theorems include rings, commutative C^*-algebras, Kakutani's (M)-spaces, vector lattices, divisible Abelian lattice-ordered groups.

For our work, the most relevant structures are the ones used by Yosida.

(For the following definition, think of $C(X, \mathbb{R})$.)

Definition (Riesz, 1928)

A *vector lattice* (a.k.a. Riesz space) is a real vector space equipped with a lattice order such that

1. (Translation invariance) if $x \leqslant y$ then $x + z \leqslant y + z$.
2. (Positive Homogeneity) if $x \leqslant y$ then $\lambda x \leqslant \lambda y$ for any real scalar $\lambda \geqslant 0$.

For Yosida's duality additional conditions on a vector lattice are needed.

Definition

A (*strong order*) *unit* of a vector lattice V is an element $1 \in V$ such that, for any $x \in V$ there is n such that $x \leqslant n1$ (where $n1 \coloneqq \underbrace{1 + \cdots + 1}_{n \text{ times}}$).

A *unital vector lattice* is a vector lattice with a designated unit.

(Example: in $C(X, \mathbb{R})$, the constant function 1 is a unit (using compactness of X).)

A unital vector lattice can be equipped with a pseudometric induced by its unit 1:

$$d(v, w) \coloneqq \texttt{inf}\,\{\lambda \in \mathbb{R} \mid \lambda \geqslant 0 \text{ and } |v - w| \leqslant \lambda 1\},$$

where $|x| \coloneqq x \vee -x$ is the absolute value of x.

In $C(X, \mathbb{R})$, this pseudometric is the uniform metric.

The pseudometric is a metric iff the vector lattice is *Archimedean*, i.e. for all $x, y \in V$, if for all $n \in \mathbb{N}$ $nx \leqslant y$, then $x \leqslant 0$.

Theorem (Yosida duality, 1941)

KHaus *is dually equivalent to the category of Archimedean metrically complete unital vector lattices.*

Every Archimedean metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for a unique compact Hausdorff space X.

In this work we extend Yosida's duality by replacing vector lattices with Abelian lattice-ordered group.

Definition

An *Abelian lattice-ordered group* (*Abelian ℓ-group*, for short) is an abelian group equipped with a lattice order such that $x \leqslant y$ implies $x + z \leqslant y + z$ (translation invariance).

Much of the theory of vector lattices goes through for Abelian ℓ-groups.

- In contrast with vector lattices, Abelian lattice-ordered group have a **finitary language** (because we do not have multiplication by all real scalars).
- Abelian ℓ-groups are widely used in the study of **MV-algebras**, the algebraic semantics of Łukasiewicz many-valued logic.

A *unit* of an Abelian ℓ-group is defined as in the vector lattice case (for any x there is n such that $x \leqslant n1$). For the pseudometric induced by the unit:

$$d(v, w) \coloneqq \mathtt{inf}\left\{\frac{p}{q} \in \mathbb{Q} \mid p \geqslant 0, q > 0, \text{ and } q|v - w| \leqslant p1\right\}.$$

The notion of being *Archimedean* is defined as in the vector lattice case ($(\forall n \in \mathbb{N}\; nx \leqslant y) \Rightarrow x \leqslant 0$), and it is equivalent to the above pseudometric being a metric.

Functional representation

Archimedean metrically complete unital Abelian ℓ-groups enjoy the following representation theorem.

Theorem (Goodearl-Handelman 1980)

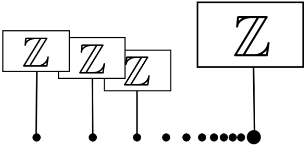
- *Let X be a compact Hausdorff space. For each $x \in X$ let A_x be either $A_x = \mathbb{R}$ or $A_x = \frac{1}{n}\mathbb{Z}$. Then, the algebra of functions*

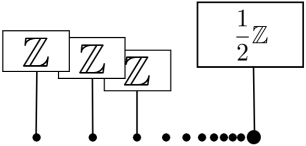
$$\{f : X \to \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\},$$

  is an Archimedean metrically complete unital Abelian ℓ-group.
- *Every Archimedean metrically complete unital Abelian ℓ-group can be represented in this way.*

Not a 1:1 correspondence

This is not a 1:1 correspondence; two “non-isomorphic labelled spaces” may give isomorphic Archimedean metrically complete unital Abelian ℓ-group.



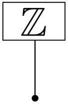


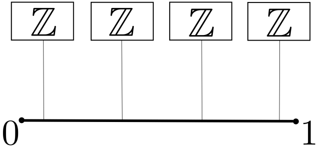
In both cases

$$\left\{f : X \to \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\right\} =$$

$$= \{f : X \to \mathbb{Z} \mid f \text{ is definitely constant}\}.$$

Not a 1:1 correspondence





In both cases

$$\{f : X \to \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\} =$$

$$= \{f : X \to \mathbb{Z} \mid f \text{ is constant}\} \cong \mathbb{Z}.$$

Our aim

Our aim: make the Goodearl-Handelman representation into a **categorical duality**.

We can encode a label A_x of Goodearl-Handelman via a natural number.

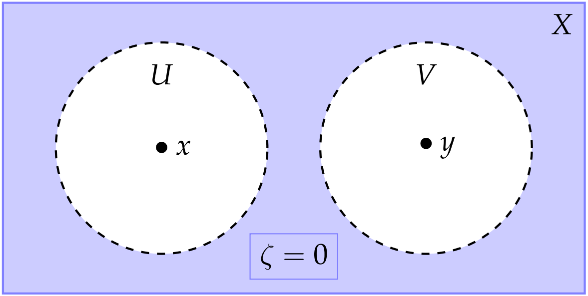
$$\begin{array}{ccc} \frac{1}{n}\mathbb{Z} & \rightsquigarrow & n. \\ \mathbb{R} & \rightsquigarrow & 0. \end{array}$$

Instead of a system of labels $(A_x)_{x\in X}$ we have a function $\zeta\colon X \to \mathbb{N}$.

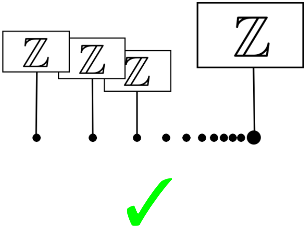
Definition

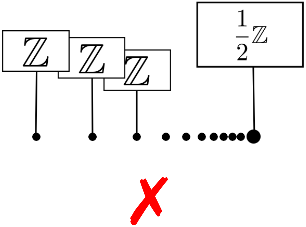
An *a-normal space* (for 'arithmetically normal space') is a pair (X, ζ) where X is a compact Hausdorff space and $\zeta \colon X \to \mathbb{N}$ is a function such that

1. (Continuity) For every $n \in \mathbb{N}$, $\zeta^{-1}[\operatorname{div} n]$ is closed.
2. (Separation) For any two distinct points x and y of X, there exist two disjoint open neighbourhoods U and V of x and y, respectively, such that, for every $z \in X \setminus (U \cup V)$, $\zeta(z) = 0$.



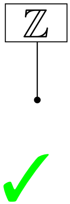
Examples

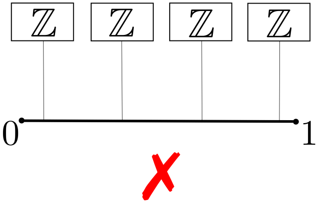




In the second example, $\zeta^{-1}[\mathrm{div}\, 1]$ $(= \zeta^{-1}[1])$ is not closed.

Examples





In the second example, the elements $0, 1 \in [0, 1]$ cannot be separated by opens U and V so that $\zeta(x) = 0$ for all $x \in [0, 1] \setminus (U \cup V)$.

Main result

Theorem

*The category of **a-normal spaces** is dually equivalent to the category of **Archimedean metrically complete unital Abelian ℓ-groups**.*

Thank you for listening!