A generalization of de Vries duality to closed relations between compact Hausdorff spaces

Marco Abbadini
University of Salerno, Italy

Joint work with G. Bezhanishvili and L. Carai

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Stone duality

Stone duality

Stone representation theorem for Boolean algebras [Stone, 1936] drew a connection between syntax and semantics: each formula of classical propositional logic is interpreted as a set of possible worlds, where

- logical "or" $\leftrightarrow$ union of sets of worlds,
- logical "and" $\leftrightarrow$ intersection of sets of worlds,
- logical "negation" $\leftrightarrow$ complementation of set of worlds.

Stone duality

Stone's representation theorem is a duality (= dual categorical equivalence) between the category of Boolean algebras (syntax) and the category of Stone spaces (a.k.a. profinite spaces, or Boolean spaces), i.e. compact Hausdorff spaces with a basis of closed open sets (semantics).

$$\mathbf{BA} \cong \mathbf{Stone}^{\mathrm{op}}.$$

More information (quotient of algebras $A \twoheadrightarrow B$)

=

fewer possible worlds (inclusion of spaces $X_A \hookleftarrow X_B$).

Topology $\longleftrightarrow$ Logic

Stone duality showed how to use topology to do logic.

In the other direction, one can obtain a logical calculus to investigate topological spaces of interest.

One of the spaces of most interest in real-life applications is $\mathbb{R}$ with its Euclidean topology.

Consider a rod: What can we say for certain about its length ℓ? We cannot tell that $\ell = 20$ cm. Any measurement will have finite precision: Maybe we measure 20.1 ± 0.3 cm, so that we know $\ell \in (19.8, 20.4)$. But we can never be sure it is exactly 20 cm long.

(It is possible to verify that $x \in Y$) iff $x \in \mathrm{int}(Y)$.

Slogan: open sets are semidecidable properties.

1. for every value of ℓ, either $\ell \in (19, 21)$ or $\ell \notin (19, 21)$.
2. it is not true that, for every value of ℓ, we can verify that $\ell \in (19, 21)$ or we can verify that $\ell \notin (19, 21)$.
3. For every value of ℓ we can verify $\ell \in (19, 21)$ or we can verify $\ell \notin [19.1, 20.9]$. This uses the facts that $[19.1, 20.9] \subseteq (19, 21)$, $[19.1, 20.9]$ is closed and $(19, 21)$ is open.

For $S, T \subseteq \mathbb{R}$:

$$\begin{aligned} \mathrm{cl}(S) \subseteq \mathrm{int}(T) &\iff \text{for all } x \text{ we can verify } x \in T \text{ or we can verify } x \notin S \\ &\iff \text{"we can verify } T \text{ or refute } S\text{".} \end{aligned}$$

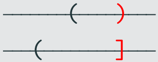
We write $S \prec T$ (and we say that S is <u>well-inside</u> T) to mean $\mathrm{cl}(S) \subseteq \mathrm{int}(T)$.

Example

$(-1, 1) \prec (-2, 2]$.

Non-example

$(0, 1) \not\prec (-1, 1]$.



We should think of $S \prec T$ as a strict version of containment ($S \subseteq T$), or of entailment/implication.

Some of the best behaved spaces are compact Hausdorff spaces: these are the closed subsets of some power of $[0,1]$. Just like Stone spaces are the closed subsets of $\{0,1\}$.

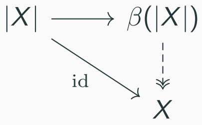
Compact Hausdorff spaces are related to my research because my background is in many-valued logic, where one takes $[0,1]$ as the set of truth values instead of $\{0,1\}$.

[Duskin, 1969]: the category of compact Hausdorff spaces is dual to a variety of infinitary algebras, which is finitely axiomatizable [Marra, Reggio, 2017].

[A., Reggio, 2020]: the category of Nachbin's compact ordered spaces ($=$ a generalization of Priestley spaces) is dual to a variety of infinitary algebras, which is finitely axiomatizable [A., 2021]. This is related to a many-valued positive logic.

The compact Hausdorff space X can be recovered from the Boolean algebra $\mathcal{P}(X)$ together with the proximity $\prec$ defined by $S \prec T$ iff $\mathrm{cl}(S) \subseteq \mathrm{int}(T)$.

We construct the Stone space Y dual to the Boolean algebra $\mathcal{P}(X)$. (In this case: $Y = \beta(|X|)$ is the Stone-Čech compactification of X with discrete topology.) We associate to $\prec$ a closed equivalence relation E on Y. (This requires an extension of Stone duality to closed relations.) We get X back as the quotient Y/E.



The space X is represented by $(\mathcal{P}(X), \prec)$.

$\mathcal{P}([0,1])$ is hard to treat computationally.

However, there are different possibilities to represent $[0,1]$ with a pair $(B, \prec)$ via the procedure

$$(B, \prec) \to (Y, E) \to Y/E \cong [0,1].$$

Take the following countable Boolean algebra A of subsets of $[0,1]$: the elements of A are $[a_1, b_1) \cup \cdots \cup [a_n, b_n)$ with rational endpoints (with the prescription that you add also 1 if $b_n = 1$.) The relation $\prec$ is defined by $A \prec B$ iff $\text{cl}(A) \subseteq \text{int}(B)$.

Construct the Stone space Y dual of A, and consider the closed equivalence relation E on Y that corresponds to $\prec$; the space Y/E is isomorphic to $[0,1]$.

So, $[0,1]$ can be represented via a countable Boolean algebra together with a binary relation on it.

Yet another procedure is to use regular open sets.

A regular open subset of a space X is an open subset U of X such that $U = \text{int}(\text{cl}(U))$.

Example

$(0,1) \cup (2,3)$ is a regular open subset of $\mathbb{R}$, while $(-1,0) \cup (0,1)$ is not.

For any space X, the set $\mathrm{RO}(X)$ of regular open subsets of X is a complete boolean algebra with respect to the inclusion order [MacNeille, 1937], [Tarski, 1937].

$$\begin{aligned} A \wedge B &= A \cap B; \\ A \vee B &= \mathrm{int}(\mathrm{cl}(A \cup B)); \\ 0 &= \emptyset; \\ 1 &= X; \\ \neg A &= \mathrm{int}(X \setminus A). \end{aligned}$$

Let X be a compact Hausdorff space. Consider the well-inside relation $\prec$ on $\mathrm{RO}(X)$:

$$A \prec B \iff \mathrm{cl}(A) \subseteq B.$$

Then from $(\mathrm{RO}(X), \prec)$ we can construct a Stone dual (Y, E), and the quotient Y/E is homeomorphic to X. (Y is the so-called Gleason cover of X.)

This construction was considered by de Vries in an algebraic approach to compactifications [de Vries, 1962].

The correspondence between compact Hausdorff spaces and pairs $(B, \prec)$ can be turned into a duality.

Definition

A proximity Boolean algebra is a boolean algebra equipped with a binary relation $\prec$ (called proximity) s.t., for all a, b, b_1, b_2:

1. $a' \leq a \prec b \leq b'$ implies $a' \prec b'$;
2. $a \prec 1$;
3. $(a \prec b_1,\ a \prec b_2)$ implies $a \prec b_1 \wedge b_2$;
4. $a \prec b$ implies $a \leq b$;
5. $a \prec b$ implies that there exists c such that $a \prec c \prec b$.
6. $a \prec b$ implies $\neg b \prec \neg a$;

($\prec$ can be thought of as a version of the entailment relation that satisfies weaker axioms.)

Given a continuous function $f\colon X \to Y$, I define a relation S_f from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ by setting $A \; S_f \; B$ iff $f[\mathsf{cl}(A)] \subseteq \mathsf{int}(B)$.

When $X = Y$ and $F = \mathrm{id}$, $S_f = \prec$.

Definition

A relation $S\colon A \to B$ between proximity Boolean algebras is called a functional compatible subordination if

1. S is a subordination:
   1.1 $0 \; S \; b$;
   1.2 $a \; S \; 1$;
   1.3 if $a_1 \; S \; b$ and $a_2 \; S \; b$, then $(a_1 \vee a_2) \; S \; b$;
   1.4 if $a \; S \; b_1$ and $a \; S \; b_2$, then $a \; S \; (b_1 \wedge b_2)$;
   1.5 if $a' \leq a \; S \; b \leq b'$, then $a' \; S \; b'$;
2. S is compatible (with the relations $\prec_A$ and $\prec_B$):

$$a \; S \; b \iff \exists \; a' \in A : a \prec_A a' \; S \; b \iff \exists b' \in B : a \; S \; b' \prec_B b;$$

3. S is functional:
   3.1 if $a \; S \; 0$, then $a = 0$;
   3.2 if $b_1 \prec_B b_2$, then there is $a \in A$ s.t. $\neg a \; S \; \neg b_1$ and $a \; S \; b_2$.

$\mathbf{ProxBA^{FS}}$:= category of proximity Boolean algebras and functional compatible subordinations. Composition = composition of relations.

Given relations $X \xrightarrow{R} Y \xrightarrow{S} Z$, their composite $S \circ R \colon X \to Z$ is defined by

$$x \,(S \circ R)\, z \iff \exists y \in Y \text{ s.t. } x \,R\, y \,S\, z.$$

$\mathbf{KHaus^{F}}$:= category of compact Hausdorff spaces and continuous functions.

Theorem

The categories $\mathbf{KHaus^{F}}$ *and* $\mathbf{ProxBA^{FS}}$ *are equivalent.*

Equivalence vs duality: a matter of taste: slightly modifying the functionality axioms one obtains a duality.

Definition

$\mathbf{KHaus}^{\mathbf{R}}$:= category

- whose objects are compact Hausdorff spaces, and
- whose morphisms from X to Y are the closed relations $R \subseteq X \times Y$ (equivalently, those relations $R\colon X \to Y$ such that the R-image of a closed subset of X is closed and the R-preimage of a closed subset of Y is closed).

Composition of morphisms is the usual composition of relations.

$\mathbf{ProxBA^S}$:= category of proximity Boolean algebras and compatible subordinations. Composition = composition of relations.

Main Theorem

$\mathbf{KHaus^R}$ and $\mathbf{ProxBA^S}$ are equivalent.

(We obtained the previous one as a corollary of this.)

In $\mathbf{ProxBA^S}$ there are isomorphic objects that are quite different. They might even fail to share the same cardinality.

If one wants to avoid this, one can select, for each compact Hausdorff space, the pair $(\mathsf{RO}(X), \prec)$ and all of its structure-preserving-isomorphic copies.

Definition ([de Vries, 1962])

A de Vries algebra is a proximity boolean algebra that is complete and satisfies

$$a \neq 0 \Rightarrow \exists b \neq 0 : b \prec a.$$

For every compact Hausdorff space X, $(\mathrm{RO}(X), \prec)$ is a de Vries algebra.

For every de Vries algebra A there exists a compact Hausdorff space X (unique up to homeomorphism) such that A is structure-preserving-isomorphic to $(\mathrm{RO}(X), \prec)$ [de Vries, 1962].

[de Vries, 1962]: $\mathbf{KHaus^F}$ is dual to a category whose objects are de Vries algebras and whose morphisms are functions satisfying certain properties. However, composition was not function composition, which is a major drawback.

$\mathbf{DeV^{FS}}$:= category of de Vries algebras and functional compatible subordinations (composition = composition of relations).

$$\mathbf{DeV^{FS}} \subseteq \mathbf{ProxBA^{FS}}$$

$\mathbf{DeV^{FS}}$ is a full subcategory of $\mathbf{ProxBA^{FS}}$, and its closure under iso is $\mathbf{ProxBA^{FS}}$.

Theorem

$\mathbf{KHaus^{F}}$ and $\mathbf{DeV^{FS}}$ are equivalent.

Advantage over classical de Vries duality: composition of morphisms is the usual composition of relations.

$\mathbf{DeV^S}$:= category of de Vries algebras and functional compatible subordinations (composition = composition of relations).

$$\mathbf{DeV^S} \subseteq \mathbf{ProxBA^S}$$

$\mathbf{DeV^S}$ is a full subcategory of $\mathbf{ProxBA^S}$, and its closure under iso is $\mathbf{ProxBA^S}$.

Theorem

$\mathbf{KHaus^R}$ and $\mathbf{DeV^S}$ are equivalent.

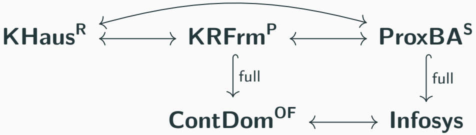
$$\begin{array}{ccccc}
\mathbf{KHaus^{F}} & \longleftrightarrow & \mathbf{ProxBA^{FS}} & \xhookleftarrow{\text{full}} & \mathbf{DeV^{FS}} \\
\big\downarrow{\scriptstyle \text{wide}} & & \big\downarrow{\scriptstyle \text{wide}} & & \big\downarrow{\scriptstyle \text{wide}} \\
\mathbf{KHaus^{R}} & \longleftrightarrow & \mathbf{ProxBA^{S}} & \xhookleftarrow{\text{full}} & \mathbf{DeV^{S}}
\end{array}$$

There are many similar results where relations were employed as morphisms. [Scott, 1982], [Larsen, Winskel, 1984], [Hoofman, 1993], [Vickers, 1993], [Abramsky, Jung, 1994], [Jung, Sünderhauf, 1996], [Jung, Kegelmann, Moshier, 2001], [Kegelmann, 2002], [Moshier, 2004].

We applied these ideas to de Vries duality.

Sketch of proof of $\mathbf{KHaus}^{\mathbf{R}} \cong \mathbf{ProxBA}^{\mathbf{S}}$

1. We apply the Karoubi envelope construction (a.k.a. splitting by idempotents or Cauchy completion) to an extension of Stone duality, where continuous functions are replaced by closed relations.
2. The equivalence $\mathbf{KHaus}^{\mathbf{R}} \cong \mathbf{ProxBA}^{\mathbf{S}}$ is the composite of
   2.1 the equivalence between $\mathbf{KHaus}^{\mathbf{R}}$ and the category $\mathbf{KRFrm}^{\mathbf{P}}$ of compact regular frames and preframe homomorphisms [Townsend, 1996], [Jung, Kegelmann, Moshier, 2001], and
   2.2 a restriction of an equivalence between the category $\mathbf{ContDom}^{\mathbf{OF}}$ of continuous domains and open filter morphisms and the category **Infosys** of continuous information systems and Lawson approximable mappings [Vickers, 1993].



To sum up

Taking certain relations (instead of functions) as morphisms between de Vries algebras solves some issues.

Furthermore, this approach allows to obtain an equivalence/duality for the category of compact Hausdorff spaces and closed relations between them.

Thank you.

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