Dualities for compact Hausdorff spaces

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Stone duality

Stone's representation theorem for Boolean algebras [Stone, 1936]:
Boolean algebras are dual to Stone spaces ($=$ compact Hausdorff spaces with a basis of closed open sets).

$$\mathbf{BA} \cong \mathbf{Stone}^{\mathrm{op}}.$$

From the late Thirties: various dualities for the category **KHaus** of compact Hausdorff spaces and continuous functions.

They associate to a compact Hausdorff space X:

1. Continuous functions from X to $\mathbb{R}$ (or $\mathbb{C}$ or $[0, 1]$).
   Motivations: represent structures of interest in functional analysis / many-valued logic.
2. Subsets of X.
   Motivations: do an algebraic approach to compactifications / obtain a logical calculus for compact Hausdorff spaces.

Dualities via subsets

The posets of open subsets of compact Hausdorff spaces can be characterized as the compact regular frames.

Theorem ([Isbell, 1972])

The category of compact Hausdorff spaces is dually equivalent to the category of compact regular frames.

Now one can obtain an equivalence for compact regular frames and compose it with Isbell's theorem to obtain a duality for compact Hausdorff spaces.

$$\mathbf{KHaus} \longleftrightarrow \mathbf{KRFrm} \longleftrightarrow ???$$

Isbell's theorem takes care of the hard work of passing from a space to a pointfree structure.

[de Vries, 1962] obtained a duality associating to $X \in \mathbf{KHaus}$ the set of regular open subsets of X. Hindsight: it factors through $\mathbf{KHaus} \leftrightarrow \mathbf{KRFrm}$.

KHaus is equivalent to a category whose object are first-order defined structures and whose morphisms are certain relations ([Moshier, 2004], specializing [Jung, Sünderhauf, 1996], building on [Smyth, 1977]).

Theorem ([A., Bezhanishvili, Carai, 2022])

KHaus *is dually equivalent to the category of S5-subordination algebras.*

An S5-subordination algebra is a pair $(B, \prec)$, where B is a Boolean algebra and $\prec$ is a binary relation on B satisfying certain first-order axioms. The morphisms of S5-subordination algebras are certain relations.

(The functors can be obtained composing the duality **KHaus** $\leftrightarrow$ **KRFrm** with a restriction of an equivalence for continuous domains [Vickers, 1993].)

Dualities via real-valued functions

Dualities for **KHaus**:

- [Gelfand, Kolmogoroff, 1939],
- [Krein, Krein, 1940],
- [Kakutani, 1941],
- [Stone, 1941],
- [Yosida, 1941],
- [Gelfand, Neumark, 1943].

[Stone, 1941]: **KHaus** is dually equivalent to the category of Archimedean metrically complete divisible lattice-ordered groups with strong unit.

[A., Marra, Spada]: duality for Archimedean metrically complete ~~divisible~~ lattice-ordered groups with strong unit.

[Cignoli, Dubuc, Mundici, 2003]: duality for locally finite MV-algebras.

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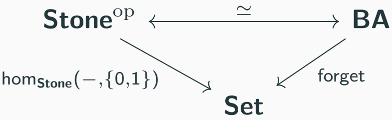
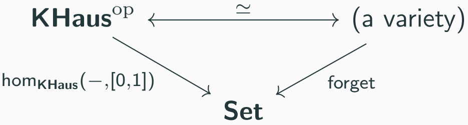


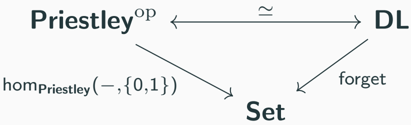


Theorem ([Duskin, 1969])

$\hom_{\mathbf{KHaus}}(-,[0,1])\colon \mathbf{KHaus}^{\mathrm{op}} \to \mathbf{Set}$ *is monadic.*

DL := category of bounded distributive lattices.

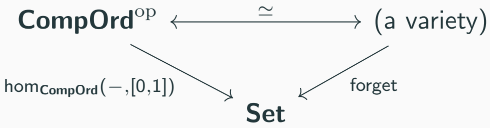
Priestley := category of Priestley spaces.



Stone sp. : compact Hausd. sp. $=$ Priestley sp. : compact ordered spaces.

Introduced by [Nachbin, 1965].

CompOrd := category of compact ordered spaces and continuous order-preserving functions.



Theorem ([A., 2019], [A., Reggio, 2020])

$\hom_{\mathbf{CompOrd}}(-,[0,1])\colon \mathbf{CompOrd}^{\mathrm{op}} \to \mathbf{Set}$ *is monadic.*

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Thank you!

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