An abstraction of the unit interval with denominators

Marco Abbadini
University of Salerno, Italy

Joint work with V. Marra and L. Spada

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Łukasiewicz logic

Łukasiewicz introduced a three-valued logic in the early 20th century.

$$\text{Set of truth values: } \{0, \tfrac{1}{2}, 1\}.$$

It was then generalized to *n*-valued (for all finite *n*) variants.

$$\text{Set of truth values: } \{0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, \tfrac{n-2}{n}, \tfrac{n-1}{n}, 1\}.$$

It was then generalized also to an infinitely many-valued variant

$$\text{Set of truth values: } [0, 1].$$

An interpretation of n-valued Łukasiewicz logic

An interpretation of n-valued Łukasiewicz logic can be given in the framework of Rényi-Ulam games, a variant of the game of Twenty Questions.

In the traditional game without lies, someone thinks of an object and another person should guess it using twenty yes-or-no questions.

In the game with lies, one is allowed to lie up to $n-2$ times.

Case $n=2$: game without lies.

Example: we should guess which letter between A, B and C Pinocchio is thinking about. Pinocchio is allowed to lie once (i.e. $n = 3$).

0: incompatible with at least two answers (thus impossible).

$\frac{1}{2}$: incompatible with exactly one answer.

1: compatible with all answers.

Possible worlds		A	B	C
		1	1	1
Are you thinking about B or C?	"No."	1	$\frac{1}{2}$	$\frac{1}{2}$
Are you thinking about C?	"Yes."	$\frac{1}{2}$	0	$\frac{1}{2}$
Are you thinking about B?	"No."	$\frac{1}{2}$	0	$\frac{1}{2}$
Are you thinking about C?	"Yes."	0	0	$\frac{1}{2}$

The letter is C.

Just like the states of knowledge of the game without lies form a Boolean algebra, the states of knowledge of the game with lies form an MV-algebra.

Boolean algebras are the algebras $\langle A, \vee, \neg, 0\rangle$ satisfying all universally quantified equation satisfied by $\{0,1\}$:

$$\forall x_1 \ \ldots \ \forall x_n \ \tau(x_1, \ldots, x_n) = \sigma(x_1, \ldots, x_n).$$

MV-algebras are the algebras $\langle A, \oplus, \neg, 0\rangle$ satisfying all universally quantified equations satisfied by $[0,1]$, where $x \oplus y = \min\{x+y, 1\}$, and $\neg x = 1 - x$.

1. $\forall x \ \ \neg\neg x$,
2. $\langle A, \oplus, 0\rangle$ is a commutative monoid,
3. $\forall x \ \ x \oplus \neg 0 = x$,
4. $\forall x \ \forall y \ \ \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

$\oplus$ is called "strong disjunction".

$x \odot y := \neg(\neg x \oplus \neg y)$.

Interpretation in $[0,1]$: $x \odot y = \max\{x + y - 1, 0\}$.

$\odot$ is called "strong conjunction".

$\oplus$ and $\odot$ are interdefinable.

Possible worlds		A	B	C
		1	1	1
Are you thinking about B or C?	"No."	1	$\frac{1}{2}$	$\frac{1}{2}$
Are you thinking about C?	"Yes."	$\frac{1}{2}$	0	$\frac{1}{2}$

$$\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(1, \frac{1}{2}, \frac{1}{2}\right) \odot \left(\frac{1}{2}, \frac{1}{2}, 1\right).$$

Examples:

- $[0, 1]$,
- $[0, 1]^2$,
- $[0, 1]^\kappa$, κ a cardinal.
- Any Boolean algebra (set $\oplus \coloneqq \vee$).
- $\{0, \frac{1}{2}, 1\}$,
- $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1\}, \quad (n \in \{1, 2, 3, \dots\})$
- $\{0, \frac{1}{2}, 1\}^{\{A,B,C\}}$,
- $\mathbb{Q} \cap [0, 1]$,
- $C(X, [0, 1]) \coloneqq \{f : X \to [0, 1] \mid f \text{ cont.}\}$. ($X$ a space)

For Boolean algebras we have a nice representation: Stone representation for Boolean algebras, a.k.a. Stone duality for Boolean algebras [Stone, 1936].

Stone duality drew a connection between syntax and semantics: each formula of classical propositional logic is interpreted as a set of possible worlds, where

- logical "or" $\leftrightarrow$ union of sets of worlds,
- logical "and" $\leftrightarrow$ intersection of sets of worlds,
- logical "negation" $\leftrightarrow$ complementation of set of worlds.

Stone duality

Stone's representation theorem is a duality (= dual categorical equivalence) between the category of Boolean algebras (syntax) and the category of Stone spaces (a.k.a. profinite spaces, or Boolean spaces), i.e. compact Hausdorff spaces with a basis of closed open sets (semantics).

More information (quotient of algebras $A \twoheadrightarrow B$)

$=$

fewer possible worlds (inclusion of spaces $X_A \hookleftarrow X_B$).

There is no equally nice representation for all MV-algebras.

So, one looks for representations of subclasses of the class of MV-algebras.

Roughly speaking, one replaces $\{0,1\}$ by $[0,1]$.

Stone spaces = closed subspaces of $\{0,1\}^\kappa$, κ a cardinal.

Compact Hausdorff spaces = closed subspaces of $[0,1]^\kappa$, κ a cardinal.

Given a Stone space X the associated Boolean algebra is the set of closed open subsets of X, that can be identified with the set

$$C(X, \{0,1\})$$

of continuous functions from X to $\{0,1\}$.

Given a compact Hausdorff space X, the set

$$C(X, [0,1])$$

of continuous functions from X to $[0,1]$ is an MV-algebra (with operations computed pointwise).

Examples:

If X is a singleton, then $C(X, [0,1]) \cong [0,1]$.

If $X = \{x_1, \ldots, x_n\}$, then $C(X, [0,1]) \cong [0,1]^n$.

The MV-algebras arising as $C(X, [0, 1])$ (for X compact Hausdorff) are precisely those that are:

1. Archimedean (i.e. there are no infinitesimals, which is equivalent to be representable as an algebra of $[0, 1]$-valued functions),
2. metrically complete (with respect to the sup metric),
3. divisible.

$\mathbb{Q} \cap [0, 1]$ is an MV-algebra that is Archimedean and divisible but not metrically complete.

$\{0, \frac{1}{2}, 1\}$ is an MV-algebra that is Archimedean, metrically complete, but not divisible.

$\{0, \frac{1}{2}, 1\}^{\{A,B,C\}}$ is an MV-algebra that is Archimedean, metrically complete, but not divisible.

We want to remove the hypothesis of divisibility, so to represent also MV-algebras such as $\{0, \frac{1}{2}, 1\}^{\{A,B,C\}}$.

$\{0, \frac{1}{2}, 1\}^{\{A,B,C\}}$ will be represented by a discrete space with three points (corresponding to A, B, C), where to each point we add the label 2 (corresponding to the denominator of $\frac{1}{2}$).

Removing “divisible” on algebras corresponds to add “denominators” on spaces.

We provide an abstraction of $[0, 1]$ that takes into account both the topology and the “denominators” of the elements of $[0, 1]$.

We define the denominator $\mathrm{den}(x)$ of each $x \in [0,1]$:

1. for $x = \frac{p}{q}$ a rational number (in its standard form), $\mathrm{den}(x) := q$.
2. for x an irrational number, $\mathrm{den}(x) := 0$.

What is an abstraction of $[0,1]$ that takes into account both the topology and the denominator map $\mathrm{den}\colon [0,1] \to \mathbb{N}$?

Compact Hausdorff spaces = closed subsets of $[0,1]^\kappa$.

??? = closed subsets of $[0,1]^\kappa$ equipped with "denominator map".

We define a denominator also for elements $(x, y) \in [0, 1]^2$.

$\mathrm{den}(x, y) = \mathrm{lcm}(\mathrm{den}(x), \mathrm{den}(y))$.

For example:

$\mathrm{den}(\frac{2}{5}, \frac{3}{5}) = 5$.

$\mathrm{den}(\frac{1}{4}, \frac{1}{3}) = 12$.

$\mathrm{den}(\frac{2}{3}, \frac{\sqrt{2}}{2}) = 0$.

$\mathrm{den}(\frac{\pi}{4}, \frac{\sqrt{2}}{2}) = 0$.

We define a denominator also for elements of $[0,1]^\kappa$.

$$\mathrm{den}((x_i)_{i\in\kappa}) = \mathrm{lcm}(\{\mathrm{den}(x_i) \mid i \in \kappa\}).$$

For example, in $[0,1]^\omega$:

$$\mathrm{den}((\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \dots)) = 4.$$

$$\mathrm{den}((\frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \dots)) = 6.$$

$$\mathrm{den}((\frac{\pi}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)) = 0.$$

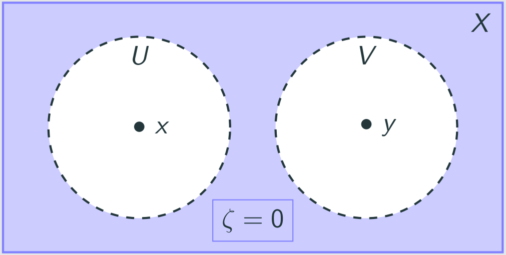
$$\mathrm{den}((\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots)) = 0.$$

How do topology and denominators interact in closed subspaces of powers of $[0,1]$?

Definition

An a-normal space (for 'arithmetically normal space') is a compact Hausdorff space X equipped with a function $\zeta\colon X \to \mathbb{N}$ s.t.

1. For every $n \in \mathbb{N}$, $\{x \in X \mid \zeta(x) \text{ divides } n\}$ is closed.
2. For distinct $x, y \in X$, there are disjoint open neighbourhoods U and V of x and y, respectively, s.t., for all $t \in X \setminus (U \cup V)$, $\zeta(t) = 0$.



A-normal spaces are the abstraction of the unit interval $[0,1]$ that takes into account both the topology and the denominator function:

Main result

Let X be a compact Hausdorff space and $\zeta\colon X \to \mathbb{N}$ a function. The following are equivalent.

1. (X, ζ) is an a-normal space.
2. There are a cardinal κ and a closed $C \subseteq [0,1]^\kappa$ such that $(X, \zeta) \cong (C, \mathrm{den})$.

Main step needed in the proof: generalization of Urysohn's lemma.

Using a-normal spaces, we represent Archimidean metrically complete MV-algebras.

Let (X, ζ) be an a-normal space.

$$\{f \colon X \to [0,1] \mid f \text{ cont.}, \forall x \in X \text{ with } \zeta(x) \neq 0, \text{ we have } f(x) \in \frac{1}{\zeta(x)}\mathbb{Z}\}$$

is an Archimedean metrically complete MV-algebra, and every Archimedean metrically complete MV-algebra can be obtained in this way.

(In fact, we have a categorical duality between a-normal spaces and Archimedean metrically complete MV-algebras.)

Thank you!

M. Abbadini, V. Marra, L. Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. Preprint at arXiv:2210.15341, 2022.