Comonadicity over Set of coalgebras for Vietoris functors

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Classical modal logic extends classical propositional logic by adding unary operators $\Diamond p$ (= it is possible that p) and $\Box p$ (= it is necessary that p), together with appropriate rules.

Algebras of classical propositional logic = Boolean algebras.

Algebras of classical modal logic = modal algebras.

A modal algebra is a Boolean algebra A with a unary operation $\Box$ s.t.

1. $\Box 1 = 1$,
2. $\forall x \forall y \;\; \Box(x \wedge y) = \Box x \wedge \Box y$.

$(\Diamond x = \neg \Box \neg x)$

All axioms are equational:

$$\forall x_1 \dots \forall x_n \;\; \tau(x_1, \dots, x_n) = \sigma(x_1, \dots, x_n).$$

Thus, modal algebras form an equational class, i.e. a variety of algebras.

Thus, the powerful techniques of universal algebra apply.

To represent modal algebras one can build on top of Stone duality.

Stone duality: Boolean algebras are dual to Stone spaces (= compact Hausdorff spaces with a basis of closed open sets).

Jónsson-Tarski duality: modal algebras are dual to *descriptive frames*, i.e. Stone spaces equipped with a binary relation R ("accessibility relation") satisfying certain properties.

One of these properties is: for every $x \in X$, $R[\{x\}]$ is closed.

So, the relation R can be alternatively described as a function from X to the set $V(X)$ of closed subsets of X satisfying certain properties.

If we topologize $V(X)$ with the Vietoris topology, R can be described as a continuous function $f\colon X \to V(X)$.

Given an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$, a coalgebra for F consists of (an object X of $\mathbf{C}$ and) a morphism $f\colon X \to F(X)$.

Morphisms of coalgebras:

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ F(X_1) & \xrightarrow{F(g)} & F(X_2) \end{array}$$

The Vietoris construction gives rise to an endofunctor $V\colon \mathbf{Stone} \to \mathbf{Stone}$.

Descriptive frame $\quad\leftrightsquigarrow\quad X \to V(X)$ continuous $\quad\leftrightsquigarrow\quad$ coalgebra for V.

The category of modal algebras is dually equivalent to the category of coalgebras for the Vietoris functor on **Stone** [Kupke, Kurz, Venema, 2004].

The Vietoris functor $V\colon \mathbf{Stone} \to \mathbf{Stone}$ extends to an endofunctor $V\colon \mathbf{KHaus} \to \mathbf{KHaus}$ on compact Hausdorff spaces.

$\mathbf{Stone}^{\mathrm{op}} \cong$ Boolean algs. $\quad$ $\mathbf{KHaus}^{\mathrm{op}} \cong$ (∞-ary) variety

[Duskin, 1969]: the representable functor

$$\hom_{\mathbf{KHaus}}(-,[0,1])\colon \mathbf{KHaus}^{\mathrm{op}} \to \mathbf{Set}$$

is monadic over **Set**.

$\mathbf{CoAlg}(V_{\mathbf{Stone}})^{\mathrm{op}} \cong$ modal algs. $\quad$ $\mathbf{CoAlg}(V_{\mathbf{KHaus}})^{\mathrm{op}} \cong$ (∞-ary) variety?

I.e.: is $\mathbf{CoAlg}(V_{\mathbf{KHaus}})^{\mathrm{op}}$ monadic over **Set**?

Given an endofunctor $F\colon \mathbf{C} \to \mathbf{C}$, we have a forgetful functor

$$\mathbf{CoAlg}(F) \to \mathbf{C}$$

that, on objects, maps $(f\colon X \to F(X))$ to X.

Strategy: show that both functors

$$\mathbf{CoAlg}(V_{\mathbf{KHaus}})^{\mathrm{op}} \to \mathbf{KHaus}^{\mathrm{op}} \xrightarrow{\hom_{\mathbf{KHaus}}(-,[0,1])} \mathbf{Set}$$

are monadic and then show that the composition is monadic.

Theorem (Elmendorf et al, 1997)

If $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{E}$ are monadic and the monad $T\colon \mathbf{D} \to \mathbf{D}$ associated to F preserves reflexive coequalizers, then the composite $G \circ F$ is monadic.

Theorem

Let $F\colon \mathbf{C} \to \mathbf{C}$ be a functor that preserves limits of ω-cochains and coreflexive equalizers. For every monadic functor $G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$, the composite $\mathbf{CoAlg}(F)^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathbf{C}^{\mathrm{op}} \xrightarrow{G} \mathbf{D}$ is monadic.

Proof.

F preserves limits of ω-cochains $\Rightarrow$ $\mathbf{CoAlg}(F)^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathbf{C}^{\mathrm{op}}$ is monadic.

Construction of the left adjoint and F preserves coreflexive equalizers $\Rightarrow$ the monad on $\mathbf{C}^{\mathrm{op}}$ preserves reflexive coequalizers. □

(There is also a version with “algebras” instead of coalgebras.)

Theorem

The opposite of the category of coalgebras for the Vietoris functor on compact Hausdorff spaces is monadic over **Set**.

Sketch of proof.

The Vietoris functor on **KHaus** preserves codirected limits [Hofmann, Neves, Nora, 2019]; thus, in particular, limits of ω-cochains.

The Vietoris functor preserves coreflexive equalizers (for pointfree proof: [Townsend, Vickers, 2014]).

The composite $\mathbf{CoAlg}(V)^{\mathrm{op}} \xrightarrow{U} \mathbf{KHaus}^{\mathrm{op}} \xrightarrow{\mathsf{hom}(-,[0,1])} \mathbf{Set}$ is monadic. □

Analogous results hold for the **lower** and **upper** Vietoris.

Analogous results hold in the setting of positive modal logic (modal logic without negation).

Positive modal algebras: bounded distributive lattices with $\Box$ and $\Diamond$ and some equational axioms.

Representation of positive modal algebras builds on Priestley duality.

Priestley duality [Priestley, 1970]: bounded distributive lattices are dual to Priestley spaces, i.e. Stone spaces with partial order and appropriate axioms.

Positive modal algebras are dual to coalgebras for the Vietoris functor on Priestley spaces.

Compact ordered space [Nachbin, 1965]: compact Hausdorff space X with a partial order that is closed in $X \times X$.

Theorem

The opposite of the category of coalgebras for the Vietoris functor on compact ordered spaces is monadic over **Set**.

(We use that the opposite of the category of compact ordered spaces is monadic over **Set** [A., Reggio, 2020].)

To sum up

The opposites of the categories of coalgebras for different versions of Vietoris functors are monadic over **Set**.

We used categorical techniques rather than algebraic/equational ones, so that we do not commit to specific signatures and equations.

Thank you!