Positive MV-algebras

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Based on:

- M. A., P. Jipsen, T. Kroupa, and S. Vannucci. *A finite axiomatization of positive MV-algebras*. Algebra Universalis, 83:28, 2022.
- M. A. *On the axiomatisability of the dual of compact ordered spaces*. PhD thesis, University of Milan, 2021. (Ch. 4)
- M. A. *Equivalence à la Mundici for commutative lattice-ordered monoids*. Algebra Universalis, 82:45, 2021.

Łukasiewicz logic

Łukasiewicz logic (Łukasiewicz, 1920, Łukasiewicz, Tarski, 1930):
$[0, 1]$ as the set of truth values.

MV-algebras

Algebraic semantics of classical propositional logic = Boolean algebras.

Algebraic semantics of Łukasiewicz logic = MV-algebras (Chang, 1958).

Consider $[0,1]$ with the operations:

- $x \oplus y := \min\{x + y, 1\}$.
  Example: $0.3 \oplus 0.2 = 0.5$ but $0.7 \oplus 0.8 = 1$.
- $\neg x := 1 - x$.
  Example: $\neg 0.3 = 0.7$.
- 0 as a constant.

MV-algebras

Definition

An *MV-algebra* $\langle A; \oplus, \neg, 0\rangle$ is a homomorphic image of a subalgebra of a power of $\langle [0,1]; \oplus, \neg, 0\rangle$:

$$\{\text{MV-algebras}\} = \mathrm{HSP}(\langle [0,1]; \oplus, \neg, 0\rangle)$$

Equivalently, an MV-algebra is an algebra $\langle A; \oplus, \neg, 0\rangle$ satisfying all equations holding in $[0,1]$.

Theorem (Chang, 1959)

MV-algebras can be axiomatized as follows:

1. $\langle A; \oplus, 0\rangle$ *is a commutative monoid;*
2. $\neg\neg x = x$;
3. $x \oplus \neg 0 = \neg 0$;
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Examples of MV-algebras

Examples of MV-algebras.

- $\langle [0,1], \oplus, \neg, 0 \rangle$ is an MV-algebra.
- For every $n \geq 1$:

$$Ł_n := \left\{ \frac{i}{n} \mid i \in \{0, \ldots, n\} \right\} \subseteq [0,1].$$

  For example: $Ł_2 = \{0, \frac{1}{2}, 1\}$.
- Any Boolean algebra is an MV-algebra: set $\oplus = \vee$.
- For any topological space X (e.g. an interval $[a,b] \subseteq \mathbb{R}$), the set of continuous functions from X to $[0,1]$ is an MV-algebra.

Derived MV-terms

One can then term-define:

- $1 := \neg 0$.
- $x \odot y := \neg(\neg x \oplus \neg y)$.
  In $[0,1]$: $x \odot y = \max\{x + y - 1, 0\}$.
  (Example: $0.7 \odot 0.8 = 0.5$ but $0.3 \odot 0.2 = 0$.)
- $x \vee y := (x \odot \neg y) \oplus y = (y \odot \neg x) \oplus x$.
  In $[0,1]$: $x \vee y = \max\{x, y\}$.
- $x \wedge y := (x \oplus \neg y) \odot y = (y \oplus \neg x) \odot x$.
  In $[0,1]$: $x \wedge y = \min\{x, y\}$.

If A is an MV-algebra, then $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice.

Abelian ℓ-groups

Definition

An *Abelian lattice-ordered group* (or *Abelian ℓ-group*, for short) is an Abelian group $\mathbf{G}$ equipped with a lattice order s.t., for all $x, y, z \in \mathbf{G}$,

$$x \leq y \quad \text{implies} \quad x + z \leq y + z. \qquad (\star)$$

Examples of Abelian ℓ-groups

Examples:

1. $\mathbb{R}$, with the sum.
2. If X is a topological space, then the set $C(X)$ of continuous functions from X to $\mathbb{R}$ is an Abelian ℓ-group.

MV-algebras as unit intervals

Given an Abelian ℓ-group $\mathbf{G}$ and an element $1 \in \mathbf{G}$ that is *positive* (i.e. $1 \geq 0$), the set

$$\Gamma(\mathbf{G}, 1) := \{x \in G \mid 0 \leq x \leq 1\}$$

is an MV-algebra with

- $x \oplus y := (x + y) \wedge 1$,
- $\neg x := 1 - x$.
- 0 the identity element of $\mathbf{G}$.

Theorem (Mundici, 1986)

Every MV-algebra arises in this way.

For example: $[0, 1] = \Gamma(\mathbb{R}, 1)$.

Mundici's equivalence

Definition

A *strong unit* of an Abelian ℓ-group $\mathbf{G}$ is a positive element $1 \in \mathbf{G}$ s.t. for all $x \in \mathbf{G}$ there is $n \in \mathbb{N}_{>0}$ s.t.

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Theorem (Mundici, 1986)

The categories

1. *of Abelian ℓ-groups with strong unit and unit-preserving homomorphisms, and*
2. *of MV-algebras and homomorphisms*

are equivalent.

Positive MV-algebras

Bounded distributive lattices as subreducts

Relationship between bounded distributive lattices and Boolean algebras?

Bounded distr. lattices $= \{\vee, \wedge, 0, 1\}$-subreducts of Boolean algebras.

$\vee$, $\wedge$, 0, 1 are order-preserving, and they term-generate all order-preserving Boolean terms.

Bounded distributive lattices = positive subreducts of Boolean algebras.

Positive MV-algebras

Definition

Positive MV-algebras $:= \{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras.

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 are order-preserving in each coordinate. We leave out $\neg$, which is not order-preserving.

Theorem (Cintula, Kroupa, 2013)

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 *generate all order-preserving terms of MV-algebras.*

Positive MV-algebras

Bounded distributive lattices = positive subreducts of Boolean algebras.

Positive MV-algebras = positive subreducts of MV-algebras.

$$\frac{\textbf{Positive MV-algebras}}{\text{MV-algebras}} = \frac{\text{Bounded distributive lattices}}{\text{Boolean algebras}}.$$

MV-algebras = many-valued version of Boolean algebras.

Positive MV-algebras = many-valued version of bounded distrib. lattices.

Examples of positive MV-algebras

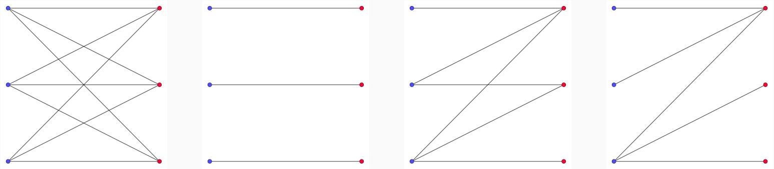
Examples of positive MV-algebras:

1. Every MV-algebra, such as $[0,1]$, or $Ł_n$.
2. Every bounded distributive lattice (set $\oplus := \vee$ and $\odot := \wedge$).
3. Given an ordered topological space X (e.g. an interval $[a,b] \subseteq \mathbb{R}$), the set of continuous order-preserving functions from X to $[0,1]$ is a positive MV-algebra.

Examples of positive MV-algebras

Positive (subdirect) subreducts $A \leq Ł_2 \times Ł_2$:

1. Full product: $Ł_2 \times Ł_2$.
2. Diagonal: $\{(a,a) \mid a \in Ł_2\} = \{(0,0), (\frac{1}{2}, \frac{1}{2}), (1,1)\}$.
3. Order-preserving functions: $\{(a_1, a_2) \in Ł_2 \times Ł_2 \mid a_1 \leq a_2\} = \{(0,0), (0, \frac{1}{2}), (0,1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1), (1,1)\}$.
4. Ordinal sum: $\{(a_1, a_2) \in Ł_2 \times Ł_2 \mid a_1 = 0 \text{ or } a_2 = 1\} = \{(0,0), (0, \frac{1}{2}), (0,1), (\frac{1}{2}, 1), (1,1)\}$.
5. Those obtained from 3. and 4. by swapping the two coordinates.



Sketch of main results

Main results:

1. Finite axiomatization.
2. Positive MV-algebras = unit intervals of certain lattice-ordered monoids.

Finite axiomatization of positive MV-algebras

Axiomatization of positive MV-algebras

Positive MV-algebras cannot be axiomatized by equations (they are not closed under homomorphic images).

Positive MV-algebras form a quasi-variety (generated by $[0, 1]$).

Axiomatization of positive MV-algebras

Theorem [A., Jipsen, Kroupa, Vannucci, 2022]

Positive MV-algebras are axiomatized by:

1. $\langle A; \oplus, 0 \rangle$ and $\langle A; \odot, 1 \rangle$ are commutative monoids;
2. $\langle A; \vee, \wedge \rangle$ is a distributive lattice;
3. Both $\oplus$ and $\odot$ distribute over both $\vee$ and $\wedge$;
4. $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$;
5. $(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z$;
6. $(x \oplus y) \odot z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \wedge z$;
7. If $x \oplus z = y \oplus z$ and $x \odot z = y \odot z$, then $x = y$.

In $[0, 1]$, both sides of (4) equal $\min\{\max\{x + y + z - 1, 0\}, 1\}$.

Finitely many quasi-equations.

Equivalence with certain lattice-ordered monoids

Unit intervals

MV-algebras = intervals of Abelian ℓ-groups.

Positive MV-algebras = intervals of certain lattice-ordered monoids.

Definition

A *commutative distributive ℓ-monoid* is a commutative monoid equipped with a distributive lattice-order s.t. $+$ distributes over $\vee$ and $\wedge$, i.e.

$$x + (y \vee z) = (x + y) \vee (x + z),$$

$$x + (y \wedge z) = (x + y) \wedge (x + z).$$

A commutative distributive ℓ-monoid is said to be *cancellative* if

$$x + z = y + z \quad \text{implies} \quad x = y.$$

Examples of cancellative commutative distributive ℓ-monoids:

- $\mathbb{R}$.
- Every Abelian ℓ-group.
- Given an ordered topological space X (such as an interval $[a, b] \subseteq \mathbb{R}$), the set of continuous order-preserving functions from X to $\mathbb{R}$.

Lattice-ordered monoids

Given a cancellative commutative distributive ℓ-monoid $\mathbf{M}$ and a positive invertible element $1 \in \mathbf{M}$, the set

$$\Gamma(\mathbf{M}, 1) \coloneqq \{x \in \mathbf{M} \mid 0 \leq x \leq 1\}$$

is a positive MV-algebra, with

- $x \oplus y \coloneqq (x + y) \wedge 1$;
- $x \odot y \coloneqq (x + y - 1) \vee 0$;
- $\vee$, $\wedge$, 0, 1 as in $\mathbf{M}$.

Theorem (A., Jipsen, Kroupa, Vannucci, 2022)

Every positive MV-algebra arises in this way.

Examples

Examples:

- $[0,1] \cong \Gamma(\mathbb{R}, 1)$.
- $\{0,1\} \cong \Gamma(\mathbb{Z}, 1)$.
- The three-element bounded distributive lattice, as a positive MV-algebra (set $\oplus := \vee$ and $\odot := \wedge$), is isomorphic to

$$\Gamma(\{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b\}, (1,1)) = \{(0,0) < (0,1) < (1,1)\}.$$

Equivalence à la Mundici for positive MV-algebras

Definition

A *strong unit* of a (cancellative) commutative distributive ℓ-monoid **M** is a positive invertible element $1 \in \mathbf{M}$ s.t., for every $x \in \mathbf{M}$, there is $n \in \mathbb{N}_{>0}$ s.t.

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Theorem (A., Jipsen, Kroupa, Vannucci, 2022)

The categories

1. of cancellative commutative distributive ℓ-monoids with strong unit and unit-preserving homomorphisms, and
2. of positive MV-algebras and homomorphisms

are equivalent.

Beyond cancellation

Equivalences à la Mundici

Abelian ℓ-groups with 1 $\cong$ MV-algebras

cancellative commut. distr. ℓ-monoids with 1 $\cong$ Positive MV-algebras

commut. distr. ℓ-monoids with 1 $\cong$???

MV-monoidal algebras

Definition (A., 2021)

A *MV-monoidal algebra* is an algebra $\langle A; \oplus, \odot, \vee, \wedge, 0, 1\rangle$ s.t.

1. $\langle A; \oplus, 0\rangle$ and $\langle A; \odot, 1\rangle$ are commutative monoids;
2. $\langle A; \vee, \wedge\rangle$ is a distributive lattice;
3. Both $\oplus$ and $\odot$ distribute over both $\vee$ and $\wedge$;
4. $(x \oplus y) \odot ((x \odot y) \oplus z) = (x \odot (y \oplus z)) \oplus (y \odot z)$;
5. $(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z$;
6. $(x \oplus y) \odot z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \wedge z$.

We removed

$$\text{If } x \oplus z = y \oplus z \text{ and } x \odot z = y \odot z, \text{ then } x = y.$$

Finitely many equations.

Equivalence à la Mundici for ℓ-monoids

MV-monoidal algebras are precisely the unit intervals of commutative distributive ℓ-monoids.

Theorem

The categories

1. of commutative distributive ℓ-monoids with strong unit and unit-preserving homomorphisms, and
2. of MV-monoidal algebras and homomorphisms

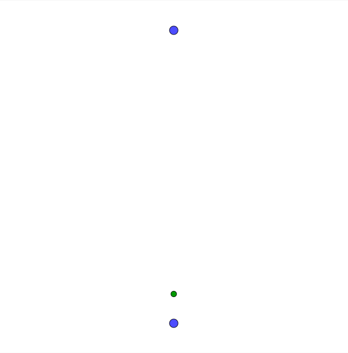
are equivalent.

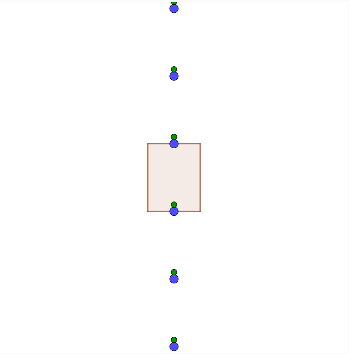
Examples of MV-monoidal algebras

1. Every positive MV-algebra.
2. $\{\mathbf{0} < \varepsilon < \mathbf{1}\}$ with $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = \mathbf{0}$. This is $\Gamma(\mathbf{M}, 1)$, where

   $$\mathbf{M} = \{\dots -\mathbf{1} < -1 + \varepsilon \quad < \mathbf{0} < \varepsilon \quad < \mathbf{1} < 1 + \varepsilon \quad < \mathbf{2} < 2 + \varepsilon \dots\}$$

   with $\varepsilon + \varepsilon = \varepsilon$. E.g.: $(2 + \varepsilon) + (3 + \varepsilon) = 5 + \varepsilon$.
   $\mathbf{M}$ is not cancellative. $\{\mathbf{0} < \varepsilon < \mathbf{1}\}$ is not a positive MV-algebra.





Free MV-extension

For every bounded distributive lattice L there is an essentially unique embedding into a Boolean algebra.

Theorem

For every bounded distributive lattice L, for all injective bounded lattice homomorphisms $f\colon L \hookrightarrow A$ and $g\colon L \hookrightarrow B$ into Boolean algebras, the Boolean algebras generated by the images of f and g are isomorphic over L.

In other words: if L is a bounded distributive lattice, B is a Boolean algebra, $\iota\colon L \hookrightarrow B$ is an injective bounded lattice homomorphism and the image of ι generates B, then the embedding ι is free (i.e. it is the unit of the left adjoint to the forgetful functor $\mathrm{BA} \to \mathrm{BDL}$).

The same thing happens for positive MV-algebras.

Theorem

For every positive MV-algebra L, for all injective bounded lattice homomorphisms $f\colon L \hookrightarrow A$ and $g\colon L \hookrightarrow B$ into MV-algebras, the MV-algebras generated by the images of f and g are isomorphic over L.

This is equivalent to the fact that every MV-equation is equivalent (in MV-algebras) to a system of equations in the positive fragment. E.g.: for all x, y, z in an MV-algebra, we have

$$x = \neg y \iff \begin{cases} x \odot y = 0; \\ x \oplus y = 1. \end{cases}$$

$$x \oplus \neg y = z \iff \begin{cases} x \wedge y = z \odot y; \\ 1 = z \oplus y. \end{cases}$$

Recap

Recap

Definition

Positive MV-algebras $:=$ positive subreducts of MV-algebras.

- Positive MV-algebras have a finite quasi-equational axiomatization.
- Positive MV-algebras are precisely the unit intervals of cancellative commutative distributive ℓ-monoids.
- Beyond cancellation: the unit intervals of commutative distributive ℓ-monoids are MV-monoidal algebras (axiomatized by finitely many equations).
- The embedding of a positive MV-algebra into some MV-algebra is essentially unique.

Thank you!