Positive MV-algebras

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Positivity

A *positive* review: ✓.

A *positive* viral test: ✗.

A *positive* MV-algebra: ?

This talk: provide some initial results for positive MV-algebras that shall help to develop their theory.

Positive MV-algebras

Positive fragment of Łukasiewicz m.v. prop. logic		Positive fragment of classical propositional logic
$\downarrow$		$\downarrow$
$\dfrac{\textbf{Positive MV-algebras}}{\text{MV-algebras}}$	$=$	$\dfrac{\text{Bounded distributive lattices}}{\text{Boolean algebras}}$
$\uparrow$		$\uparrow$
Łukasiewicz many-valued propositional logic ($[0, 1]$ as set of truth values)		Classical propositional logic

Positive algebras

Positive modal algebras (Dunn, 1995) $:=$ "modal algebras without negation".

Positive relation algebras $:=$ "relation algebras without complementation".

Positive MV-algebras

(Cintula, Kroupa, 2013): positive fragment of Łukasiewicz logic in game theory.

(Cabrer, Jipsen, Kroupa, 2019): defined positive MV-algebras, provided some initial results.

Duality for Nachbin's compact ordered spaces ($\cong$ stably compact spaces).

Just like Heyting algebras are based on bounded distributive lattices, an appropriate *many-valued* version of Heyting algebras (MV-intuitionistic logic) may be based on positive MV-algebras.

Well-behaved study case for some results in duality theory and general algebra.

Łukasiewicz logic

Łukasiewicz (many-valued propositional) logic [Łukasiewicz, 1920, Łukasiewicz, Tarski, 1930]:
$[0, 1]$ as the set of truth values.

Algebraic semantics

Algebraic semantics of Łukasiewicz logic = MV-algebras (Chang, 1958).

Consider $[0,1]$ with the operations:

- $x \oplus y := \min\{x+y, 1\}$.
  Example: $0.3 \oplus 0.2 = 0.5$, and $0.7 \oplus 0.8 = 1$.
- $\neg x := 1 - x$.
  Example: $\neg 0.3 = 0.7$.
- 0 as a constant.

MV-algebras

Definition (Chang, 1958, Mangani, 1973)

An *MV-algebra* (for *Many-Valued algebra*) is an algebra $\langle A; \oplus, \neg, 0\rangle$ s.t.

1. $\langle A; \oplus, 0\rangle$ is a commutative monoid;
2. $\neg\neg x = x$;
3. $x \oplus \neg 0 = \neg 0$;
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV-algebras are the algebras $\langle A; \oplus, \neg, 0\rangle$ satisfying all equations holding in $[0, 1]$, i.e.:

Theorem (Chang, 1959)

$[0, 1]$ *generates the variety of MV-algebras.*

In other words: $\{\text{MV-algebras}\} = \mathrm{HSP}([0, 1])$.

Examples of MV-algebras

Examples of MV-algebras:

- $[0, 1]$.
- Subalgebras of $[0, 1]$, such as:
  1. $\{0, 1\}$ (here, $\oplus = \vee$);
  2. $Ł_2 := \{0, \frac{1}{2}, 1\}$.
- Any Boolean algebra: set $\oplus = \vee$.
- The set $[0, 1]^X$ of functions from a set X to $[0, 1]$.

Derived MV-terms

Every MV-algebra has a bounded distributive lattice reduct:

- $x \vee y := \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.
- $x \wedge y := \neg(\neg x \vee \neg y)$.
- $1 := \neg 0$.

The corresponding order on $[0,1]$ is the usual order. On $[0,1]^X$ is the pointwise order.

One can also term-define the De Morgan dual of $\oplus$:

- $x \odot y := \neg(\neg x \oplus \neg y)$.
  In $[0,1]$: $x \odot y = \max\{x + y - 1, 0\}$.
  Example: $0.7 \odot 0.8 = 0.5$, and $0.3 \odot 0.2 = 0$.

Abelian lattice-ordered groups

MV-algebras can be understood as intervals of Abelian lattice-ordered groups.

Definition

An *Abelian lattice-ordered group* (or *Abelian ℓ-group*, for short) is an Abelian group $\mathbf{G}$ equipped with a lattice order s.t.:

(Translation invariance) for all $x, y, z \in \mathbf{G}$, $x \leq y$ implies $x + z \leq y + z$.

Examples:

1. $\mathbb{R}$, with the sum.
2. The set $\mathbb{R}^X$ of functions from a set X to $\mathbb{R}$.

MV-algebras as unit intervals

For **G** an Abelian ℓ-group and $1 \in \mathbf{G}$ *positive* (i.e. $1 \geq 0$), the interval

$$\Gamma(\mathbf{G}, 1) \coloneqq \{x \in \mathbf{G} \mid 0 \leq x \leq 1\}$$

is an MV-algebra with

$$x \oplus y \coloneqq (x + y) \wedge 1, \qquad \neg x \coloneqq 1 - x, \qquad 0 \coloneqq \text{identity element of } \mathbf{G}.$$

Examples:

1. $\Gamma(\mathbb{R}, 1) = [0, 1]$,
2. $\Gamma(\mathbb{Z}, 1) = \{0, 1\}$.
3. For a set X, $\Gamma(\mathbb{R}^X, 1) = [0, 1]^X$.

Theorem (Mundici, 1986)

Every MV-algebra arises in this way, i.e. it is isomorphic to $\Gamma(\mathbf{G}, 1)$ *for some Abelian* ℓ*-group* **G** *and some positive* $1 \in \mathbf{G}$.

Positive MV-algebras

Positive MV-algebras

$$\frac{\textbf{Positive MV-algebras}}{\text{MV-algebras}} = \frac{\text{Bounded distributive lattices}}{\text{Boolean algebras}}.$$

Bounded distributive lattices as subreducts

Relationship between bounded distributive lattices and Boolean algebras?

The $\{\vee, \wedge, 0, 1\}$-reduct of any Boolean algebra is a bounded distributive lattice, as well as any subalgebra of this reduct.

$$\text{Bounded distributive lattices} = \text{subalgebras of } \{\vee, \wedge, 0, 1\}\text{-reducts of Boolean algebras.}$$

$\vee$, $\wedge$, 0, 1 are order-preserving, and they term-generate all order-preserving Boolean terms.

Bounded distributive lattices = positive subreducts of Boolean algebras.

Positive MV-algebras

Definition (Cabrer, Jipsen, Kroupa, 2019)

Positive MV-algebras $:=$ subalgebras of the $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-reducts of MV-algebras.

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 are order-preserving in each coordinate. We leave out $\neg$, which is not order-preserving.

Theorem (Cintula, Kroupa, 2013)

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 *generate all order-preserving terms of MV-algebras.*

$$\begin{aligned}\text{Positive MV-algebras} &= \text{positive subreducts of MV-algebras}\\ &= \text{many-valued version of bounded distrib. lattices.}\end{aligned}$$

Examples of positive MV-algebras

Examples of positive MV-algebras:

- Every MV-algebra, such as $[0,1]$, $\{0,1\}$, $Ł_2$.
- Every bounded distributive lattice (set $\oplus := \vee$ and $\odot := \wedge$).

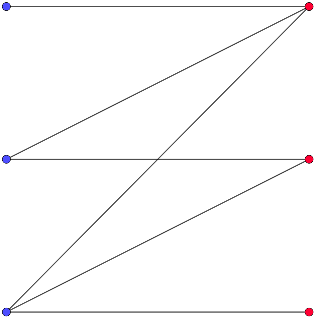
Positive MV-algebras are a common generalization of MV-algebras and bounded distributive lattices.

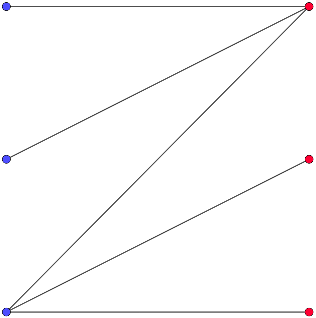
- For a poset X, the set of order-preserving functions from X to $[0,1]$.

Examples of positive MV-algebras

Some subreducts of the MV-algebra $Ł_2 \times Ł_2 = \{0, \frac{1}{2}, 1\} \times \{0, \frac{1}{2}, 1\}$:

- Order-preserving functions: $\{(a, b) \in Ł_2 \times Ł_2 \mid a \leq b\}$.
- Ordinal sum: $\{(a, b) \in (Ł_2)^2 \mid a = 0 \text{ or } b = 1\}$.





Axiomatization of positive MV-algebras

Boolean algebras	**Bounded distributive lattices**	**MV-algebras**	**Positive MV-algebras**
Variety	Variety	Variety	Not variety ✗ Quasivariety ✓
Generated by $\{0,1\}$ as a quasivariety	Generated by $\{0,1\}$ as a quasivariety	Generated by $[0,1]$ as a quasivariety	Generated by $[0,1]$ as a quasivariety ✓
Finitely axiomatized	Finitely axiomatized	Finitely axiomatized	Finitely axiomatized ✓

Finite axiomatization of positive MV-algebras

Theorem (A., Jipsen, Kroupa and Vannucci, 2022)

Positive MV-algebras are axiomatized by:

1. $\langle A; \oplus, 0\rangle$ *and* $\langle A; \odot, 1\rangle$ *are commutative monoids;*
2. $\langle A; \vee, \wedge\rangle$ *is a distributive lattice;*
3. *Both* $\oplus$ *and* $\odot$ *distribute over both* $\vee$ *and* $\wedge$*;*
4. *If* $x_0 = y_0 \oplus y_1$ *and* $x_1 = y_0 \odot y_1$*, then*
   - $x_0 \oplus x_1 = x_0$ *and* $x_0 \odot x_1 = x_1$.
   - $(x_0 \odot z) \oplus x_1 = x_0 \odot (z \oplus x_1)$.
   - $x_0 \odot z = ((x_0 \odot z) \oplus x_1) \wedge z$ *and* $z \oplus x_1 = (x_0 \odot (z \oplus x_1)) \vee z$.
5. *(Cancellation) If* $x \oplus z = y \oplus z$ *and* $x \odot z = y \odot z$*, then* $x = y$.

(1–4) are equations, (5) is a quasi-equation.

Positive MV-algebras as intervals

Unit intervals

MV-algebras = intervals of Abelian lattice-ordered groups.

Positive MV-algebras = intervals of certain lattice-ordered monoids.

Lattice-ordered monoids

Definition

A *cancellative commutative distributive ℓ-monoid* is a commutative monoid equipped with a distributive lattice-order s.t. $+$ distributes over $\vee$ and $\wedge$, and

$$x + z = y + z \quad \text{implies} \quad x = y.$$

Examples of cancellative commutative distributive ℓ-monoids:

- $\mathbb{R}$, $\mathbb{Z}$, every Abelian ℓ-group.
- The set of order-preserving functions from a poset X to $\mathbb{R}$.

Lattice-ordered monoids and positive MV-algebras

Given a cancellative commutative distributive ℓ-monoid $\mathbf{M}$ and a positive invertible element $1 \in \mathbf{M}$, the set

$$\Gamma(\mathbf{M}, 1) \coloneqq \{x \in \mathbf{M} \mid 0 \leq x \leq 1\}$$

is a positive MV-algebra, with

- $x \oplus y \coloneqq (x + y) \wedge 1$;
- $x \odot y \coloneqq (x + y - 1) \vee 0$;
- $\vee$, $\wedge$, 0, 1 as in $\mathbf{M}$.

Positive MV-algebras as intervals

Theorem (A., Jipsen, Kroupa and Vannucci, 2022) ✓

Every positive MV-algebra arises in this way, i.e. it is isomorphic to $\Gamma(\mathbf{M}, 1)$ for some cancellative commutative distributive ℓ-monoid $\mathbf{M}$ and some positive invertible $1 \in \mathbf{M}$.

Examples:

- $[0, 1] \cong \Gamma(\mathbb{R}, 1)$.
- $\{0, \frac{1}{2}, 1\} \cong \Gamma(\frac{1}{2}\mathbb{Z}, 1)$.
- The three-element bounded distributive lattice, as a positive MV-algebra (set $\oplus := \vee$ and $\odot := \wedge$), is isomorphic to

$$\Gamma(\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b\}, (1, 1)) = \{(0, 0) < (0, 1) < (1, 1)\}.$$

Free MV-extension

Free MV-extension

By definition, every positive MV-algebra embeds into some MV-algebra.

Is there a canonical embedding?

Yes, for general algebraic reasons: the "universal/free map", given by the left adjoint of the forgetful functor
$\{\text{MV-algebras}\} \to \{\text{Positive MV-algebras}\}$.

Free MV-extension

Non-trivial: this is essentially the *unique* possible embedding:

Theorem (A., Jipsen, Kroupa, Vannucci, 2022)

Let L be a positive MV-algebra. Given two injective positive MV-homomorphisms

$$f\colon L \hookrightarrow A \quad \text{and} \quad g\colon L \hookrightarrow B,$$

into MV-algebras A and B, the MV-algebras generated by their images are isomorphic over L.

In particular, if the images of f and g generate A and B as MV-algebras, A and B are isomorphic over L.

Free MV-extension

Embedding + MV-generating $\Rightarrow$ Free. ✓

Given MV-algebras A and B and a partial function $f\colon L \subseteq A \to B$ such that L MV-generates A and is closed under $\oplus$, $\odot$, $\vee$, $\wedge$, 0 and 1, and $f\colon L \to B$ preserves these operations, f extends uniquely to an MV-homomorphism $A \to B$.

Free MV-extension

(For a general fact (work in progress with C. van Alten),) this is equivalent to the following: every MV-equation is equivalent (in MV-algebras) to a system of equations in the positive fragment.

Example: for all x, y, z in an MV-algebra:

$$x = \neg y \iff \begin{cases} x \odot y = 0; \\ x \oplus y = 1; \end{cases}$$

$$x \oplus \neg y = z \iff \begin{cases} x \wedge y = y \odot z; \\ y \oplus z = 1. \end{cases}$$

Recap

Recap

Definition

Positive MV-algebras := positive subreducts of MV-algebras.

1. Not a variety. ✗
2. Quasivariety, generated by $[0,1]$. ✓
3. Finite quasi-equational axiomatization. ✓
4. Intervals of certain ℓ-monoids. ✓
5. Unique embedding into MV-algebras. (Embedding + MV-generating $\Rightarrow$ Free.) ✓

Thank you!

Based on:

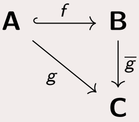
- M. A., P. Jipsen, T. Kroupa, and S. Vannucci. *A finite axiomatization of positive MV-algebras*. Algebra Universalis, 83:28, 2022.
- M. A. *On the axiomatisability of the dual of compact ordered spaces*. PhD thesis, University of Milan, 2021. (Ch. 4)
- M. A. *Equivalence à la Mundici for commutative lattice-ordered monoids*. Algebra Universalis, 82:45, 2021.

Let us fix a functional language $\mathcal{L}_+$, a sublanguage $\mathcal{L}_- \subseteq \mathcal{L}_+$, an SP-class $\mathcal{V}_+$ of algebras for $\mathcal{L}_+$ and an SP-class $\mathcal{V}_-$ of algebras for $\mathcal{L}_-$ that contains all $\mathcal{L}_-$-reducts of $\mathcal{V}_+$.

For example: $\mathcal{V}_+$ and $\mathcal{V}_-$ can be taken as the classes of Abelian groups and cancellative commutative monoids, respectively, or as the classes of Boolean algebras and bounded distributive lattices, or as the classes of groups and monoids.

Definition (Free extension property)

We say that $\mathcal{V}_+$ *has the free extension property over* $\mathcal{V}_-$ if, for all $\mathbf{A} \in \mathcal{V}_-$, all $\mathbf{B}, \mathbf{C} \in \mathcal{V}_+$, all $\mathcal{L}_-$-homomorphisms $f\colon \mathbf{A} \hookrightarrow \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{C}$ such that f is injective and the image of f $\mathcal{L}_+$-generates $\mathbf{B}$, there is a unique $\mathcal{L}_+$-homomorphism $\overline{g}\colon \mathbf{B} \to \mathbf{C}$ making the following diagram commute.



We will connect the free extension property with the following property.

Definition (Expressibility in $\mathcal{V}_-$ of equations in $\mathcal{V}_+$)

We say that *equations in $\mathcal{V}_+$ are expressible in $\mathcal{V}_-$* if, for every pair $(\sigma(x_1, \dots, x_n), \rho(x_1, \dots, x_n))$ of terms in $\mathcal{V}_+$ there is a family $\{(\alpha_i(x_1, \dots, x_n), \beta_i(x_1, \dots, x_n)) \mid i \in I\}$ of pairs of terms in $\mathcal{V}_-$ such that, for every $\mathbf{A} \in \mathcal{V}_+$ and every n-tuple $\mathbf{a} = (a_1, \dots, a_n)$ of elements of A, we have

$$[\![\sigma]\!]^{\mathbf{A}}(\mathbf{a}) = [\![\tau]\!]^{\mathbf{A}}(\mathbf{a}) \iff \text{for all } i \in I \; [\![\alpha_i]\!]^{\mathbf{A}}(\mathbf{a}) = [\![\beta_i]\!]^{\mathbf{A}}(\mathbf{a}).$$

Theorem

$\mathcal{V}_+$ has the free extension property over $\mathcal{V}_-$ if and only if equations in $\mathcal{V}_+$ are expressible in $\mathcal{V}_-$.