Jónsson-Tarski duality beyond dimension 0

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Joint work with Ivan Di Liberti.

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Modal logic

Propositional modal logic extends classical propositional logic with a unary operator $\Box p$ ($\approx$ it is necessary that p) and appropriate rules.

Modal algebras

Definition

A *modal algebra* is a Boolean algebra A with a unary operator $\Box$ s.t.

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Thus, modal algebras form an equational class, i.e. a variety of algebras.

Main question

Question

Is a "continuous version of modal logic" axiomatizable by equations?

$\{0, 1\} \to [0, 1]$.

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Modal algebras (equational)	$\overset{\text{dual}}{\longleftrightarrow}$	Descriptive frames (based on Stone spaces)
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This is a contribution to the investigation (D. Hofmann, R. Neves, P. Nora) of how much algebraic are the duals of certain categories of spaces.

Topological semantics of modal logic

Vietoris hyperspace

For a compact Hausdorff space X, the *Vietoris hyperspace of X* (Vietoris, 1922) is the set

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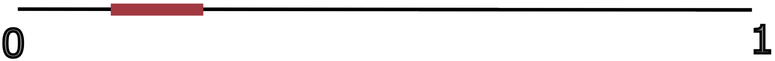

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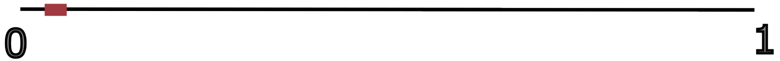

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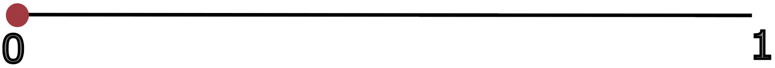

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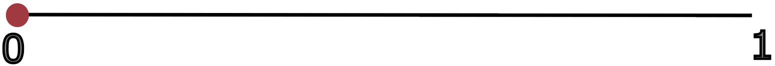

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Jónsson-Tarski duality for modal algebras

Jónsson-Tarski duality (Jónsson, Tarski, 1951; Esakia, 1974; Kupke, Kurz, Venema, 2004):

$$\text{modal algebras} \xleftrightarrow{\text{dual}} \text{descriptive frames.}$$

Descriptive frame (/*modal space*): (X, ρ) with X a Stone space and $\rho \colon X \to \mathcal{V}(X)$ a continuous function.

Stone space = compact Hausdorff space that is 0-dimensional (i.e. with a basis of closed open sets).

Beyond dimension 0

Removing 0-dimensionality

Beyond dimension 0, descriptive frames are replaced by Vietoris coalgebras.

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A *Vietoris coalgebra* ($\cong$ *modal compact Hausdorff space* (Bezhanishvili, Bezhanishvili, Harding)) is a pair (X, ρ) with X a compact Hausdorff space and $\rho \colon X \to \mathcal{V}(X)$ a continuous function.

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Example:

$$\begin{aligned} [0,1] &\longrightarrow \mathcal{V}([0,1]) \\ x &\longmapsto [x,1]. \end{aligned}$$

If $(x_i)_i \to y$ then $([x_i, 1])_i \to [y, 1]$.

Morphisms

A *Vietoris coalgebras morphism* from (X, ρ_X) to (Y, ρ_Y): continuous $g\colon X \to Y$ s.t.

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ \Big\downarrow{\scriptstyle \rho_X} & & \Big\downarrow{\scriptstyle \rho_Y} \\ \mathcal{V}(X) & \xrightarrow{g[-]} & \mathcal{V}(Y) \end{array}$$

Main result

Informally: if we allow function symbols of infinite arity, then continuous modal logic is equational.

Theorem (A., Di Liberti)

The category of Vietoris coalgebras is dually equivalent to a variety (= equational class) of infinitary algebras.

We use results of Adámek; Elmendorf, Kriz, Mandell, May; Duskin; Hofmann, Neves, Nora; Townsend, Vickers.

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    Examples:
    - $\wedge, \vee\colon [0,1]^2 \to [0,1]$ are binary function symbols.
    - Others, also of infinite arity.
  - add a unary term $\Box$.
- Add a whole bunch of equations (all equations that "hold in $[0,1]$").

Variety dual to Vietoris coalgebras

The duality maps a Vietoris coalgebra (X, ρ) to the algebra

$$C(X, [0,1]) = \{g \colon X \to [0,1] \mid g \text{ is continuous}\}$$

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- For $g \in C(X, [0,1])$,

$$\begin{aligned} \Box g\colon X &\longrightarrow [0,1] \\ x &\longmapsto \inf_{y \in \rho(x)} g(y). \end{aligned}$$

Example

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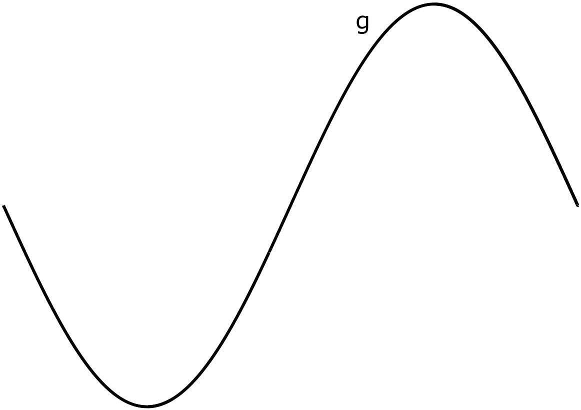

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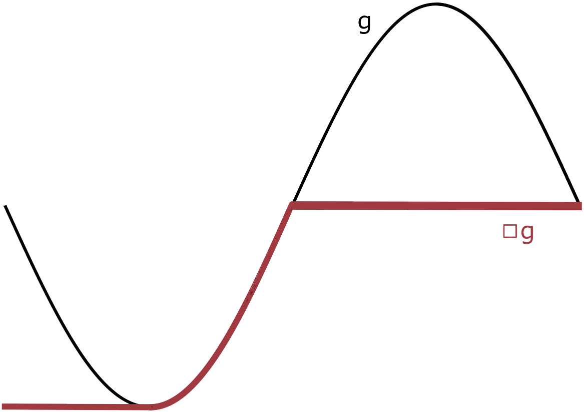

$\Box g : x \mapsto \inf_{y \geq x} g(y)$ is the greatest order-preserving function below g.

To sum up

Corollary

The category of Vietoris coalgebras is (complete, cocomplete,) Barr-coexact, and admits a regular injective regular cogenerator (the dual of the free 1-generated algebra).

Stably compact spaces

We obtained analogous results for the lower/upper/convex Vietoris hyperspaces on stably compact spaces ("continuous version of positive modal logic is equational").

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Theorem (Hofmann, Neves, Nora, 2018)

The dual of the category of coalgebras for the lower Vietoris functor on stably compact spaces is an $\aleph_1$-quasivariety.

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Thank you!

Equations of pure rank 0

Equations:

Rank 0. All equations not mentioning $\Box$ that hold in $[0,1]$.

- Equations of bounded distributive lattices for $\wedge$, $\vee$, 0, 1.
- ...

See (Marra, Reggio, 2017) for a finite base of equations.

Equations of pure rank 1

Rank 1. All equations holding in $[0, 1]$ where each variable lies in the scope of exactly one $\Box$.

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4. (For each fixed $a \in [0,1]$:) $\Box(x \oplus a) = (\Box x) \oplus a$.