Free extensions

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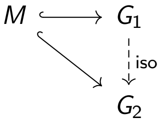
Abelian groups and commutative monoids

Fact

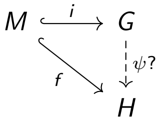
For each commutative monoid M, there is (up to isomorphism) at most one embedding $M \hookrightarrow G$ of M into an Abelian group G whose image generates G as a group.

Uniqueness up to isomorphism means:



Example: $\mathbb{N} \hookrightarrow \mathbb{Z}$ (wrt addition).

Sketch of proof. Let i and f be injective monoid homomorphisms with generating images. Wlog, $i\colon M \subseteq G$.



Every $z \in G$ is a difference $z = x - y$ of elements of M.

Set $\psi(z) := f(x) - f(y)$.

It is well-defined:

$$
\begin{aligned}
x - y = x' - y' &\iff x + y' = x' + y \\
&\implies f(x + y') = f(x' + y) \\
&\iff f(x) + f(y') = f(x') + f(y) \\
&\iff f(x) - f(y) = f(x') - f(y').
\end{aligned}
$$

Further, one proves that ψ is a group isomorphism that extends f.

Key fact used

For every Abelian group G and all $x, y, x', y' \in G$,

$$x - y = x' - y' \iff x + y' = x' + y.$$

Fact

Any equation between two terms in the language of Abelian groups is equivalent to an equation in the language of monoids.

Example: $-x + y - z = x \iff y = x + x + z$.

Groups and monoids

For arbitrary (i.e. non-necessarily commutative) monoids/groups, we do **not** have this uniqueness of embeddings.

E.g.: the monoid $\{x, y, z\}^*$ of words on three letters has two non-isomorphic generating embeddings into groups:

- into the free group $\mathrm{Free}(\{x, y, z\})$

$$\begin{aligned} \{x, y, z\}^* &\longrightarrow \mathrm{Free}(\{z, y, z\}) \\ x &\longmapsto x \\ y &\longmapsto y \\ z &\longmapsto z. \end{aligned}$$

- into the free group $\mathrm{Free}(\{x, y\})$

$$\begin{aligned} \{x, y, z\}^* &\longrightarrow \mathrm{Free}(\{x, y\}) \\ x &\longmapsto x \\ y &\longmapsto y \\ z &\longmapsto xy^{-1}x. \end{aligned}$$

Groups and monoids

For groups, the equation $z = xy^{-1}x$ cannot be expressed via an equation in the language of monoids.

Main result

Our main result is an equivalence between two conditions: the "unique embeddability property" and the "expressibility property".

Recall that a *prevariety* (a.k.a. *SP-class*) is a class of similar algebras closed under subalgebras and products.

Examples: any variety, any quasi-variety.

Setting

- An algebraic language $\mathcal{L}_+$ and a sublanguage $\mathcal{L}_- \subseteq \mathcal{L}_+$.
- Two prevarieties $\mathcal{V}_+$ and $\mathcal{V}_-$ for $\mathcal{L}_+$ and $\mathcal{L}_-$ respectively, such that all $\mathcal{L}_-$-reducts of algebras in $\mathcal{V}_+$ belong to $\mathcal{V}_-$.

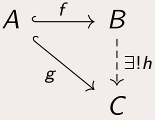
For example:

1. $\mathcal{V}_+ = \{\text{Abelian groups}\}$, $\mathcal{V}_- = \{\text{commutative monoids}\}$.
2. $\mathcal{V}_+ = \{\text{groups}\}$, $\mathcal{V}_- = \{\text{monoids}\}$.

Recall: Every commutative monoid has at most one embedding into an Abelian group with generating image.

Definition

Unique embeddability property := for all $A \in \mathcal{V}_-$, $B, C \in \mathcal{V}_+$ and injective $\mathcal{V}_-$-homomorpisms $f\colon A \hookrightarrow B$ and $g\colon A \hookrightarrow C$ whose images $\mathcal{V}_+$-generate B and C respectively, there is a $\mathcal{V}_+$-isomorphism $h\colon B \to C$ making the following diagram commute.



For $\mathcal{V}_+ = \{\text{Abelian groups}\}$ and $\mathcal{V}_- = \{\text{commutative monoids}\}$: the unique embeddability property holds.

For $\mathcal{V}_+ = \{\text{groups}\}$ and $\mathcal{V}_- = \{\text{monoids}\}$: the unique embeddability property does not hold.

An equivalent formulation is: every injective $\mathcal{V}_-$-homomorpism $A \hookrightarrow B$ ($A \in \mathcal{V}_-$, $B \in \mathcal{V}_+$) whose image $\mathcal{V}_+$-generate B is a *free extension*, i.e. the free way in which A embeds into an algebra in $\mathcal{V}_+$. In other words, it is (up to an iso) the component at A of the unit of the adjunction that has the forgetful functor $\mathcal{V}_+ \to \mathcal{V}_-$ as a right adjoint.

In short: "injective + generating $\Rightarrow$ free".

Definition

Expressibility property := every equation in $\mathcal{L}_+$ is equivalent to a system of equations in $\mathcal{L}_-$.
I.e.: for each pair $(\sigma(x_1, \ldots, x_n), \rho(x_1, \ldots, x_n))$ of terms in $\mathcal{L}_+$, there is a (finite) set of pairs $(\alpha_i(x_1, \ldots, x_n), \beta_i(x_1, \ldots, x_n))_i$ of terms in $\mathcal{L}_-$ s.t., for all $A \in \mathcal{V}_+$ and $x_1, \ldots, x_n \in A$,

$$\sigma(x_1, \ldots, x_n) = \rho(x_1, \ldots, x_n) \;\Leftrightarrow\; \forall i \;\; \alpha_i(x_1, \ldots, x_n) = \beta_i(x_1, \ldots, x_n).$$

For $\mathcal{V}_+ = \{\text{Abelian groups}\}$ and $\mathcal{V}_- = \{\text{commutative monoids}\}$: it holds. E.g. $x - y = x' - y'$ iff $x + y' = x' + y$.

For $\mathcal{V}_+ = \{\text{groups}\}$ and $\mathcal{V}_- = \{\text{monoids}\}$: it does not hold. E.g.: $z = xy^{-1}x$.

Main theorem

Unique embeddability property $\iff$ expressibility property.

Examples

Rational vector spaces and Abelian groups

$\mathcal{V}_+ := \{\text{rational vector spaces}\}$, $\mathcal{V}_- := \{\text{Abelian groups}\}$.

True or false? (Unique embeddability property)

For each Abelian group G, there is (up to isomorphism) at most one embedding (of Abelian groups) $G \hookrightarrow V$ into a rational vector space whose image generates V (as a rational vector space).

(Example: $\mathbb{Z} \hookrightarrow \mathbb{Q}$.)

Expressibility property? $\frac{1}{2}x + y = \frac{2}{3}z$ is equivalent to $3x + 6y = 4z$.

Fact (Expressibility property)

Every equation in the language of $\mathbb{Q}$-vector spaces is equivalent to an equation in the language of Abelian groups.

Answer: true.

Complex and real vector spaces

$\mathcal{V}_+ = \{$complex vector spaces$\}$, $\mathcal{V}_- = \{$real vector spaces$\}$.

True or false? (Unique embeddability property)

For each real vector space V, there is (up to isomorphism) at most one embedding (of Abelian groups) $V \hookrightarrow W$ into a complex vector space whose image generates W (as a complex vector space).

Expressibility property? The equation $x = iy$ does **not** seem to be equivalent to a system of equations in the language of real vector spaces.

The embeddings $f\colon \mathbb{R} \times \mathbb{R} \hookrightarrow \mathbb{C} \times \mathbb{C}$ and $g\colon \mathbb{R} \times \mathbb{R} \hookrightarrow \mathbb{C}$ are non-isomorphic. $f(0,1) \neq if(1,0)$, $g(0,1) = ig(1,0)$.

Answer: false.

In turn, we can then infer that the equation $x = iy$ is not equivalent to an equation in the language of real vector space.

Abelian ℓ-groups and monoids

$\mathcal{V}_+ =$ Abelian lattice-ordered groups.

$\mathcal{V}_- =$ commutative lattice-ordered monoids.

The expressibility and the unique embeddability properties hold.

Example: $x \vee (z - y) = x - z$ is equivalent to
$(x + y + z) \vee (z + z) = x + y$.

Boolean algebras and lattices

$\mathcal{V}_+ = \{\text{Boolean algebras}\}$, $\mathcal{V}_- = \{\text{bounded distributive lattices}\}$.

Expressibility property

Every equation in the language of Boolean algebras is equivalent to a system of equations in the language of bounded distributive lattices.

Example: $x \vee \neg y = \neg z$ is equivalent to $\begin{cases} x \wedge z = 0; \\ x \vee y = x \vee z. \end{cases}$

Unique embeddability property

For each bounded distributive lattice L, there is (up to isomorphism) at most one embedding $L \hookrightarrow B$ into a Boolean algebra whose image generates B (as a Boolean algebra).

This embedding is called the *free Boolean extension* of L.

MV-algebras and positive MV-algebras

$\mathcal{V}_+ =$ MV-algebras.

$\mathcal{V}_- =$ positive MV-algebras (i.e. $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras).

Both the expressibility and the unique embeddability properties hold.

Modal algebras and positive modal algebras

$\mathcal{V}_+ =$ Modal algebras.

$\mathcal{V}_- =$ Positive modal algebras.

The unique embeddability and the expressibility properties do *not* hold.

Example: the equation $\Diamond(x \wedge \neg y) = 0$ is not expressible via a system of equations in the language of positive modal algebras.

Summing it up and future directions

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Main theorem

Unique embeddability property $\Longleftrightarrow$ expressibility property.

This was established in the setting of prevarieties.

Future direction (from a suggestion of a referee): does the equivalence extend beyond algebraic structures to a general model-theoretic setting?

Thank you!