Soft sheaf representations in Barr-exact categories

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We generalize a sheaf-theoretic result from varieties of finitary algebras to Barr-exact categories (with some additional conditions).

Soft sheaves of varieties of algebras

From varieties to Barr-exact categories

- 1940s: sheaves introduced.
- 1960s: applications of sheaves to rings and modules (Grothendieck, Dauns & Hofmann, Pierce, . . .)
- 1970s: sheaf representations of general algebraic structures (Comer, Cornish, Davey, Keimel, Wolf, . . .)

A distributive lattice of commuting congruences
of an algebraic structure $\mathbf{A}$ yields a sheaf representation of $\mathbf{A}$.

Two congruences R and S **commute** (or **permute**) if $R \circ S = S \circ R$.

$\mathrm{Con}(\mathbf{A}) := \{\text{congruences of } \mathbf{A}\}$. (Ordered by inclusion.)

Theorem (Wolf, 1974)

Let $\mathbf{A}$ *be a finitary algebra, and let* $\mathbf{L}$ *be a bounded distributive sublattice of* $\mathrm{Con}(\mathbf{A})$ *s.t. every two congruences in* $\mathbf{L}$ *commute. Then* $\mathbf{A}$ *is the algebra of global sections of a sheaf on the space of prime ideals of* $\mathbf{L}$*.*

- If $\mathbf{A}$ is congruence-distributive and congruence-permutable, then we can take $L = \mathrm{Con}(\mathbf{A})$.
- If $\mathbf{A}$ is congruence-distributive, congruence-permutable and has the compact intersection property (= finitely generated congruences are closed under finite meets) (e.g.: Boolean algebras), then we can take

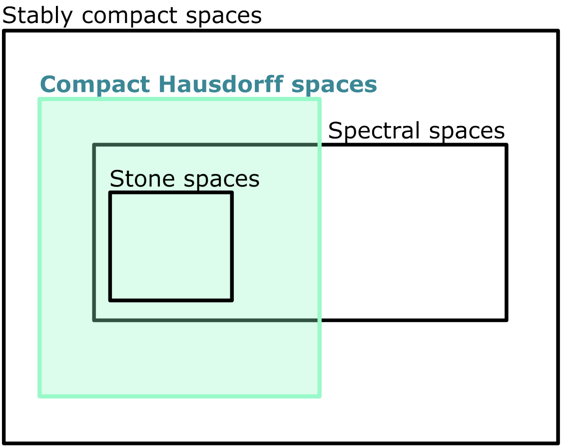
$$\mathbf{L} = \{\text{finitely generated congruences}\}.$$

$\Rightarrow$ Space of prime ideals of $\mathbf{L} \cong$ space of meet-prime congruences of $\mathbf{A}$.
The stalks are finitely subdirectly irreducible (and hence "quite simple").
For Boolean algebras: $\rightarrow$ Stone duality.

Congruence-permutability $\leftrightsquigarrow$ gluing of two compatible partial sections.

Gehrke and van Gool (2018): which sheaf representations correspond to distributive lattices of pairwise commuting congruences? The “soft” ones.

We phrase the results for compact Hausdorff spaces, but in fact they hold for stably compact spaces.



(Wolf $\to$ spectral spaces. Gehrke and van Gool $\to$ stably compact spaces.)

Definition ($\sim$ Godement, 1958)

A sheaf $\Omega(X)^{\mathrm{op}} \to \mathsf{Set}$ on a compact Hausdorff space X is **soft** if every local section on a compact subset of X can be extended to a global section.

Example: the sheaf

$$\begin{aligned} \Omega([0,1])^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ U &\longmapsto C(U, \mathbb{R}) \end{aligned}$$

of continuous real-valued functions on $[0,1]$ is soft. E.g.: a section on $[\frac{1}{3}, \frac{2}{3}]$ is (roughly speaking) consists of a continuous $[\frac{1}{3}, \frac{2}{3}] \to \mathbb{R}$ together with its local behaviour at $\frac{1}{3}^{-}$ and $\frac{2}{3}^{+}$. It can be extended to a continuous function $[0,1] \to \mathbb{R}$.

$\mathcal{K}(X) := \{$compact subsets of $X\}$. (Ordered by inclusion.)

A **sheaf representation of A on X** is a sheaf F on X s.t. $F(X) \cong \mathbf{A}$.

Theorem (Gehrke & van Gool, 2018)

Let X be a compact Hausdorff space and $\mathbf{A}$ a nonempty finitary algebra in a variety. There is a bijection between:

1. *isomorphism classes of soft sheaf representations of $\mathbf{A}$ on X;*
2. *$(\wedge, \bigvee)$-preserving maps $\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Con}(\mathbf{A})$ with image consisting of pairwise commuting congruences.*

$1 \rightsquigarrow 2$. To a soft sheaf representation $F\colon \Omega^{\mathrm{op}}(X) \to \mathbf{A}$ we associate

$$\begin{aligned} \mathcal{K}(X)^{\mathrm{op}} &\longrightarrow \mathsf{Con}(\mathbf{A}) \\ K &\longmapsto \ker(F(X) \twoheadrightarrow F(K)). \end{aligned}$$

$2 \rightsquigarrow 1$. To $\rho\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Con}(\mathbf{A})$ we associate

$$\begin{aligned} \Omega(X)^{\mathrm{op}} &\longrightarrow \mathcal{V} \\ U &\longmapsto \lim_{K \in \mathcal{K}(X),\, K \subseteq U} \mathbf{A}/\rho(K). \end{aligned}$$

Soft sheaves of varieties of algebras

From varieties to Barr-exact categories

Variety of finitary algebras

$\downarrow$

Complete cocomplete Barr-exact category
in which directed colimits commute with finite limits

Examples: varieties of finitary algebras, Grothendieck toposes.

Theorem (Vitale, 1998)

A category is equivalent to a **localization of an algebraic category** *if and only if it is exact and has a regular generator which admits all copowers, and directed colimits exist and commute with finite limits.*

Recall that a Set-valued sheaf is **soft** if every local section on a compact subspace extends to a global section. I.e., if the restriction map from global sections to local sections on a compact subspace is surjective.

Definition

A C-valued sheaf $F\colon \Omega(X)^{\mathrm{op}} \to \mathsf{C}$ on a compact Hausdorff space X is **soft** if for every compact $K \subseteq X$ the morphism

$$F(X) \to \operatorname*{colim}_{U \in \Omega(X)\colon K \subseteq U} F(U)$$

is a regular epimorphism.

$\mathsf{Equiv}(A) :=$ poset of internal equivalence relations on A.

Two internal equivalence relations R and S **commute** if $R \circ S = S \circ R$.

Theorem (A. & Reggio, 2023)

Let C *be a complete cocomplete Barr-exact category in which directed colimits commute with finite limits. Let* $A \in \mathsf{C}$ *be such that* $A \to 1$ *is a regular epi. Let* X *be a compact Hausdorff space. There is a bijection between:*

1. *isomorphism classes of soft sheaf representations of* A *on* X*;*
2. $(\wedge, \bigvee)$*-preserving maps* $\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Equiv}(A)$ *with image consisting of pairwise commuting internal equivalence relations.*

$\Omega(X)^{\mathrm{op}} \to \mathsf{C}$	$\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Equiv}(A)$
representation of A	preserves bottom
gluing on empty family	preserves top
gluing on binary family	preserves binary meets, binary joins, and the image consists of ker-commuting elements
gluing on directed family	preserves directed joins

To prove the theorem, we first proved a version for regular categories (rather than Barr-exact ones). In this version, the notion of commuting internal equivalence relations is replaced by an appropriate notion of commuting quotients.

Quotient of an object A $:=$ isomorphism class of a regular epimorphism with domain A.

$$\begin{array}{ccc} A & \overset{f}{\twoheadrightarrow} & B \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \sim} \\ & & C. \end{array}$$

$\mathrm{Quo}(A)$ $:=$ poset of quotients of A.
A quotient $g\colon A \twoheadrightarrow C$ is *smaller* than $f\colon A \twoheadrightarrow B$ if there is $h\colon B \to C$ s.t.

$$g \leq f \ : \qquad \begin{array}{ccc} A & \overset{f}{\twoheadrightarrow} & B \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \exists h} \\ & & C. \end{array}$$

Definition

Two quotients $A \twoheadrightarrow B$ and $A \twoheadrightarrow C$ of an object A are **ker-commuting** if their kernel pairs commute and their composition is effective.

When C is Barr-exact, $\mathrm{Quo}(A)^{\mathrm{op}} \cong \mathrm{Equiv}(A)$, and two quotients are ker-commuting iff their kernel pairs commute.

Theorem (A. & Reggio, 2023)

Let C be a complete cocomplete regular category where directed colimits commute with finite limits, let $A \in \mathsf{C}$ be such that $A \to 1$ is a regular epi and let X be a compact Hausdorff space. There is a bijection between:

1. *isomorphism classes of soft sheaf representations of A on X;*
2. *$(\bigwedge, \vee)$-preserving maps $\mathcal{K}(X) \to \mathrm{Quo}(A)$ with image consisting of pairwise ker-commuting quotients.*

When C is Barr-exact, (2) is also equivalent to

2. $(\wedge, \bigvee)$-preserving maps $\mathcal{K}(X)^{\mathrm{op}} \to \mathrm{Equiv}(A)$ with image consisting of pairwise commuting internal equivalence relations,

yielding the theorem in a previous slide.

Our result holds also when X is a stably compact space (replace "compact" by "compact saturated").

Also pointfree: replace $\Omega(X)$ with a stably continuous lattice.

We did not consider sites.

To sum up

A distributive lattice of commuting congruences of a finitary algebraic structure **A** yields a sheaf representation of **A** (Wolf, 1974).

The sheaves arising in this way are the soft ones (Gehrke, van Gool, 2018).

We generalized this from varieties of finitary algebras to complete cocomplete Barr-exact categories in which directed colimits commute with finite limits.

M. Abbadini, L. Reggio.
Barr-exact categories and soft sheaf representations.
Journal of Pure and Applied Algebra, 227(12):107413 (2023).

Thank you!