An abstraction of the unit interval with denominators

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Outline

1. We provide an abstraction of $[0, 1]$ with the Euclidean topology and the "denominator map" $[0, 1] \to \mathbb{N}$.
2. Application: representation of certain ordered groups.

Part 1:
An abstraction of the unit interval with denominators

$$\begin{aligned} \mathrm{den}\colon [0,1] &\longrightarrow \mathbb{N} \\ x &\longmapsto \mathrm{den}(x) \coloneqq \begin{cases} q & \text{if } x = \frac{p}{q} \in \mathbb{Q}, \text{with } p \text{ and } q \text{ coprime,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \end{aligned}$$

It is convenient to identify the codomain $\mathbb{N}$ of den with the set $\mathbf{N}$ of topologically closed additive subgroups of $\mathbb{R}$ containing 1:

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbf{N} \\ n &\longmapsto \begin{cases} \frac{1}{n}\mathbb{Z} & \text{if } n \neq 0 \\ \mathbb{R} & \text{if } n = 0. \end{cases} \end{aligned}$$

Then, we identify the denominator map $\mathrm{den}\colon [0,1] \to \mathbb{N}$ with

$$\begin{aligned} [0,1] &\longrightarrow \mathbf{N} \\ x &\longrightarrow \text{closure of the additive subgroup of } \mathbb{R} \text{ generated by 1 and } x \\ &= \begin{cases} \frac{1}{q}\mathbb{Z} & \text{if } x = \frac{p}{q} \in \mathbb{Q}, \text{ with } p \text{ and } q \text{ coprime,} \\ \\ \mathbb{R} & \text{if } x \text{ is irrational.} \end{cases} \end{aligned}$$

Closed subs. of $[0,1]^\kappa$	$=$	Comp. Hausd. spaces.
Closed subs. of $[0,1]^\kappa$, with denominators	$=$	???

$$\begin{aligned}\operatorname{den}\colon [0,1]^2 &\longrightarrow \mathbb{N}\\ (x,y) &\longmapsto \operatorname{lcm}(\operatorname{den}(x),\operatorname{den}(y)).\end{aligned}$$

$$\operatorname{den}\left(\frac{1}{4},\frac{1}{3}\right) = 12;$$

$$\operatorname{den}\left(\frac{2}{5},\frac{3}{5}\right) = 5;$$

$$\operatorname{den}\left(\frac{\pi}{4},\frac{\sqrt{2}}{2}\right) = 0;$$

$$\operatorname{den}\left(\frac{2}{3},\frac{\sqrt{2}}{2}\right) = 0.$$

$$\begin{aligned}\operatorname{den}\colon [0,1]^2 &\longrightarrow \mathbf{N}\\ (x,y) &\longmapsto \text{closure of the subgroup of } \mathbb{R} \text{ generated by } 1,\ x \text{ and } y.\end{aligned}$$

We define a denominator for elements of $[0,1]^\kappa$:

$$\begin{aligned}\mathrm{den}\colon [0,1]^\kappa &\longrightarrow \mathbb{N}\\ (x_i)_{i\in\kappa} &\longmapsto \mathrm{lcm}(\{\mathrm{den}(x_i) \mid i \in \kappa\}).\end{aligned}$$

$$\mathrm{den}\left(\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4},\dots\right) = 4;$$

$$\mathrm{den}\left(\frac{1}{3},\frac{1}{2},\frac{1}{3},\frac{1}{2},\dots\right) = 6;$$

$$\mathrm{den}\left(\frac{\pi}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots\right) = 0;$$

$$\mathrm{den}\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\dots\right) = 0; .\mathbb{R}.$$

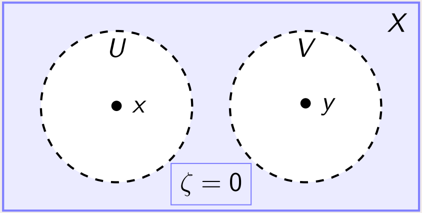
$$\begin{aligned}\mathrm{den}\colon [0,1]^\kappa &\longrightarrow \mathbf{N}\\ (x_i)_{i\in\kappa} &\longmapsto \text{closure of the subgr. of } \mathbb{R} \text{ generat. by 1 and all } x_i\ (i \in \kappa).\end{aligned}$$

How do topology and denominators interact in closed subspaces of powers of $[0, 1]$?

Definition

A *compact a-Hausdorff a-space* (for "compact arithmetically Hausdorff arithmetic space") is a pair (X, ζ) where X is a topological space and $\zeta\colon X \to \mathbb{N}$ a function s.t.

1. (Arithmetic Continuity) For all $n \in \mathbb{N} \setminus \{0\}$,
   $\{x \in X \mid \zeta(x) \text{ divides } n\}$ is closed.
2. (Compactness) X is compact.
3. (Arithmetic Hausdorffness) For all $x \neq y \in X$, there are disjoint opens $U \ni x$ and $V \ni y$ s.t., for all $t \in X \setminus (U \cup V)$, $\zeta(t) = 0$.

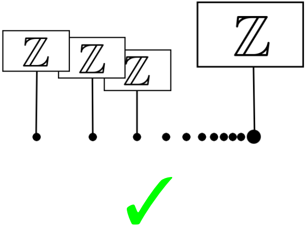


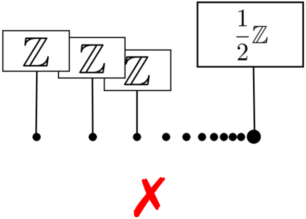
$([0,1], \mathrm{den})$ is a compact a-Hausdorff a-space.

Also, any of its powers.

Examples

Compact a-Hausdorff a-spaces?

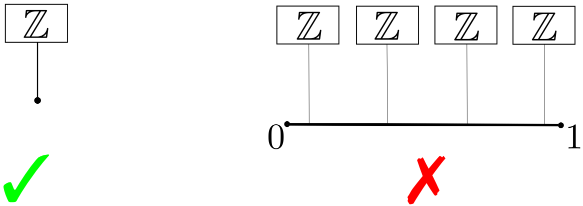




In the second example we do not have arithmetic continuity because the set $\{x \in X \mid \zeta(x) \subseteq \mathbb{Z}\}$ is not closed.

Examples

Compact a-Hausdorff a-spaces?



In the second example we do not have arithmetic Hausdorfness because $0, 1 \in [0, 1]$ cannot be separated by opens U and V so that $\zeta(x) = \mathbb{R}$ for all $x \in [0, 1] \setminus (U \cup V)$.

Main result (1/2)

Compact a-Hausdorff a-spaces are the closed subsets of power of $[0,1]$:

Main result (1/2)

Let X be a topological space and $\zeta\colon X \to \mathbb{N}$ be a function. The following are equivalent.

1. (X, ζ) is a compact a-Hausdorff a-space.
2. There are a cardinal κ and a closed $C \subseteq [0,1]^\kappa$ such that $(X, \zeta) \cong (C, \mathrm{den})$.

Main step needed in the proof: generalization of Urysohn's lemma.

Part 2:
Categorical duality with certain ordered groups

We use compact a-Hausdorff a-spaces to give a representation of certain ordered groups (in the form of a categorical duality).

In the 30s and 40s, a series of papers

- [Gelfand and Kolmogorov, 1939],
- [Krein and Krein, 1940],
- [Kakutani, 1941],
- [Stone, 1941],
- [Yosida, 1941],
- [Gelfand and Neumark, 1943]

showed that various known mathematical structures can be represented by compact Hausdorff spaces via a categorical dual equivalence.

The common idea is to associate to a compact Hausdorff space X the set $C(X, \mathbb{R})$ of continuous real-valued (or complex-valued) functions on X.

The mathematical structures used in these representation theorems include rings, commutative C^*-algebras, Kakutani's (M)-spaces, vector lattices, divisible Abelian lattice-ordered groups.

For our work, the most relevant structures are the ones used by Yosida: vector lattices.

For the following definition, think of $C(X, \mathbb{R})$.

Definition (Riesz, 1928)

A *vector lattice* is a real vector space equipped with a lattice order with the following properties.

1. (Translation invariance) If $x \leq y$ then $x + z \leq y + z$.
2. (Positive Homogeneity) If $x \leq y$ then $\lambda x \leq \lambda y$ for any real scalar $\lambda \geq 0$.

For Yosida's duality a bit more structure is needed.

Definition

A *metrically complete unital vector lattice* is a vector lattice V equipped with a designated element $1 \in V$ such that the following defines a complete metric on V:

$$\mathrm{dist}(v, w) := \inf \{\lambda \in \mathbb{R}_{\geq 0} \mid -\lambda 1 \leq v - w \leq \lambda 1\},$$

Example: in $C(X, \mathbb{R})$, let 1 be the function $X \to \mathbb{R}$ constantly equal to 1; the corresponding metric is the uniform metric.

Yosida proved that every metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for a unique compact Hausdorff space X (up to homeomorphism).

Theorem (Yosida duality, 1941)

The category of compact Hausdorff spaces is dually equivalent to the category of metrically complete unital vector lattices.

One functor maps a compact Hausdorff space X to the metrically complete unital vector lattice $C(X, \mathbb{R})$.

In this work we extend Yosida's duality by replacing the structure of a vector space with the structure of an Abelian group (i.e. we lose multiplication by real scalars).

Definition

An *Abelian lattice-ordered group* is an abelian group equipped with a lattice order such that $x \leq y$ implies $x + z \leq y + z$ (translation invariance).

Abelian lattice-ordered groups are widely used in the study of *MV-algebras*, the algebraic semantics of Łukasiewicz many-valued logic.

Much of the theory of vector lattices goes through for Abelian lattice-ordered groups.

Definition

A *metrically complete unital Abelian lattice-ordered group* is an Abelian lattice-ordered group G equipped with a designated element $1 \in G$ such that the following defines a complete metric on G:

$$\mathrm{dist}(v, w) \coloneqq \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \geq 0, q > 0, \text{ and } p1 \leq q(v - w) \leq p1 \right\}.$$

Functional representation

Theorem (Representation theorem, Goodearl & Handelman 1980)

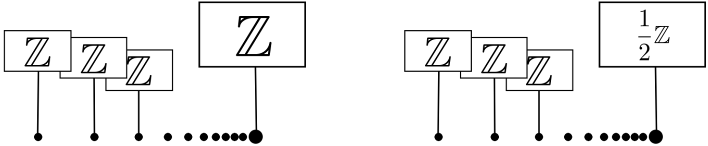
- *Let X be a compact Hausdorff space. For each $x \in X$, let D_x be either $D_x = \mathbb{R}$ or $D_x = \frac{1}{n}\mathbb{Z}$ (for some $n \in \mathbb{N} \setminus \{0\}$). Then,*

$$\left\{f \colon X \to \mathbb{R} \mid f \text{ continuous}, \forall x \in X\ f(x) \in D_x\right\}$$

  is a metrically complete unital Abelian lattice-ordered group.
- *Every metrically complete unital Abelian lattice-ordered group can be represented in this way.*

Not a 1:1 correspondence

This is not a 1:1 correspondence; two "non-isomorphic labelled spaces" may give isomorphic metrically complete unital Abelian lattice-ordered groups.

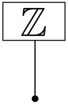


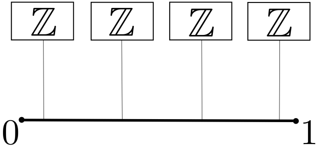
In both cases

$$\left\{f \colon X \to \mathbb{R} \mid f \text{ continuous, } \forall x \in X \ f(x) \in D_x\right\} =$$

$$= \{f \colon X \to \mathbb{Z} \mid f \text{ is definitely constant}\}.$$

Not a 1:1 correspondence





In both cases

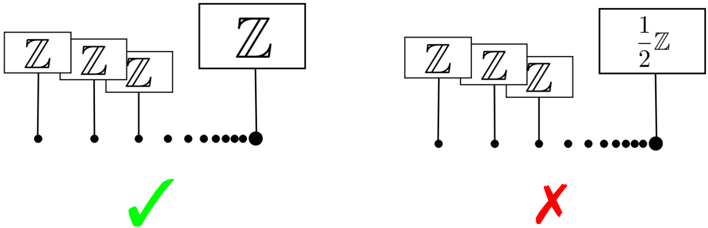
$$\left\{f : X \to \mathbb{R} \mid f \text{ continuous}, \forall x \in X\ f(x) \in D_x\right\} =$$

$$= \{f : X \to \mathbb{Z} \mid f \text{ is constant}\} \cong \mathbb{Z}.$$

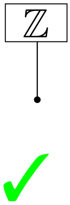
Our aim

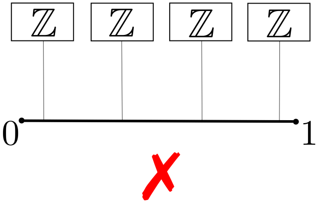
Our aim: make the Goodearl-Handelman representation into a **categorical duality**.

Compact a-Hausdorff a-spaces?



Compact a-Hausdorff a-spaces?





Main result

For every metrically complete unital Abelian lattice-ordered group G there is a unique (up to iso) compact a-Hausdorff a-space $(X, \zeta\colon X \to \mathbf{N})$ such that G is isomorphic to

$$\{f\colon X \to \mathbb{R} \mid f \text{ continuous, } \forall x \in X \; f(x) \in \zeta(x)\}.$$

Theorem

The category of **compact a-Hausdorff a-spaces** *is dually equivalent to the category of* **metrically complete unital Abelian lattice-ordered groups**.

One functor maps X to $\{f\colon X \to \mathbb{R} \mid f \text{ continuous, } \forall x \in X \; f(x) \in \zeta(x)\}$.

Recap

1. Compact a-Hausdorff a-spaces abstract $[0, 1]$ with the Euclidean topology and denominators.
2. Compact a-Hausdorff a-spaces are dual to metrically complete unital Abelian lattice-ordered groups.

M. Abbadini, V. Marra, L. Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. Preprint at arXiv:2210.15341, 2022.

Thank you!