Natural dualities

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Southern and Midlands Logic Seminar, 13 December 2023.

Stone duality

1936: Stone's representation theorem for Boolean algebras:

Every Boolean algebra is isomorphic to a field of subsets of some set.

It connects syntax and semantics: each formula of classical propositional logic is interpreted as a subset of the set of possible worlds, where

- logical "or" $\leftrightarrow$ union of sets of worlds,
- logical "and" $\leftrightarrow$ intersection,
- logical "negation" $\leftrightarrow$ complementation.

Given a Boolean algera $\mathbf{B}$, Stone equips the set $X_{\mathbf{B}}$ of homomorphisms $\mathbf{B} \to \mathbf{2}$ with an appropriate topology and shows

$$\mathbf{B} \cong \{\text{clopen } (= \text{closed} + \text{open}) \text{ subsets of } X_{\mathbf{B}}\}.$$

One-to-one correspondence between Boolean algebras and Stone spaces (a.k.a. profinite spaces or Boolean spaces).

Stone space $:=$ compact Hausdorff spaces where distinct points are separated by a clopen ($=$ closed and open) set.

E.g.: finite discrete spaces, one-point compactifications of discrete spaces.

Stone duality

In fact, Stone proved much more than a representation theorem: there is a duality (= dual categorical equivalence) between the category of Boolean algebras and the category of Stone spaces.

$$\mathsf{Bool} \cong \mathsf{Stone}^{\mathrm{op}}.$$

$$\text{More information (quotient of algebras } A \twoheadrightarrow B\text{)}$$
$$=$$
$$\text{Fewer possible worlds (inclusion of spaces } X_A \hookleftarrow X_B\text{).}$$

$$\text{Less propositions (inclusion of algebras } B \hookrightarrow A\text{)}$$
$$=$$
$$\text{Collapse of possible worlds (quotient of spaces } X_B \twoheadleftarrow X_A\text{).}$$

Some good things of Stone duality

Algebraic questions in Bool can be answered by translating them into (often simpler) questions in Stone.

1. While a coproduct $\mathbf{A} + \mathbf{B}$ of Boolean algebras is often difficult to describe, its dual, $X_{\mathbf{A}+\mathbf{B}}$, is simply the cartesian product $X_{\mathbf{A}} \times X_{\mathbf{B}}$.
2. The free Boolean algebra ($=$ Lindenbaum-Tarski algebras) on κ generators is easily described via its dual: 2^{κ}.
3. Congruences correspond to closed subspaces.
   Example: a Boolean algebra of 2^n elements has exactly 2^n congruences.

Some good things of Stone duality

The duality is logarithmic: a product $\mathbf{A} \times \mathbf{B}$ of Boolean algebras corresponds to a sum of the corresponding spaces: $X_{\mathbf{A} \times \mathbf{B}} \cong X_{\mathbf{A}} + X_{\mathbf{B}}$.

For example, the dual of a Boolean algebra of $16 = 2 \times 2 \times 2 \times 2$ elements is the discrete space of $4 = 1 + 1 + 1 + 1$ elements.

Priestley duality

[1969, Priestley]: the category of bounded distributive lattices is dually equivalent to the category of Priestley spaces.

$$\mathsf{BDL} \cong \mathsf{Priestley}^{\mathrm{op}}.$$

Priestley space $:=$ Stone space equipped with a partial order such that if $x \nleq y$ then there is a clopen upset U such that $x \in U$ and $y \notin U$.

Every bounded distributive lattice is isomorphic to the lattice of clopen upsets of a Priestley space.

New classes of algebras

To study some nonclassical logics, new classes of algebras (other than Boolean algebras) were introduced.

E.g.: Kleene (1938) introduced a three-valued logic which replaces $\{0, 1\}$ with $\{0, U, 1\}$. $\{0, U, 1\}$ is an algebra in the signature $\{0, 1, \vee, \wedge, \neg\}$.

Kleene algebras $:=$ algebras of Kleene logic $=$ algebras in the signature $\{0, 1, \vee, \wedge, \neg\}$ that satisfy each equation satisfied by $\{0, U, 1\}$:

- Bounded distributive lattices.
- (Double negation law:) $\neg\neg x = x$.
- (De Morgan laws:) $\neg(x \wedge y) = \neg x \vee \neg y$.
- $x \wedge \neg x \leq y \vee \neg y$.

In general, we do *not* have $x \wedge \neg x = 0$.

Dualities for many of these classes of algebras were found.

Benefits similar to those for Stone duality apply:

- uniform representation,
- coproducts of algebras into products of spaces,
- free algebras have an easy dual description,
- congruences correspond to closed substructures,
- logarithmic property: $X_{\mathbf{A}\times\mathbf{B}} \cong X_{\mathbf{A}} + X_{\mathbf{B}}$.

These dualities follow a similar pattern: there is a special algebra

$$\mathbf{L}$$

and the class of algebras of interest is

$$\mathbb{ISP}(\mathbf{L}) := \{\text{algebras } \textbf{i}\text{somorphic to a } \textbf{s}\text{ubalgebra of a } \textbf{p}\text{ower of } \mathbf{L}\}.$$

i.e. the class of algebras **A** (in the same language of **L**) whose elements are separated by homomorphisms from **A** to **L**.

Examples:

- Boolean algebras $= \mathbb{ISP}(\{0,1\})$ (by the Ultrafilter lemma).
- Bounded distributive lattices $= \mathbb{ISP}(\{0,1\})$.
- Kleene algebras $= \mathbb{ISP}(\{0,U,1\})$.

Logical reading: **L** is the set of truth values.

Homming to **L** then gives the functor for the duality.

Natural dualities := dualities induced by homming into a schizophrenic object.

[Davey, 1978], [Davey, Werner, 1983], [Clark, Krauss, 1984]: formulate a general theory to obtain natural dualities for $\mathbb{ISP}(\mathbf{L})$.

Davey & Werner (1983): duality for $\mathbb{ISP}(\mathbf{L})$ where $\mathbf{L}$ is a finite algebra with a near unanimity term. (E.g.: $\mathbf{L}$ has a lattice reduct, i.e. $\mathbf{L}$ has two terms $\vee$ and $\wedge$ that satisfy the axioms of lattices.)

Example: for $\mathbf{L} = \{0, 1\}$ we get Stone/Priestley dualities.

We present the restriction of this theory under some stronger hypotheses on $\mathbf{L}$:

0. $\mathbf{L}$ is finite.
1. $\mathbf{L}$ has a lattice reduct.
2. $\mathbf{L}$ has two distinct constants and every subalgebra of $\mathbf{L}$ is simple (i.e. the only congruences of a subalgebra $\mathbf{A}$ are the diagonal and $\mathbf{A} \times \mathbf{A}$).
3. For each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$.

Algebraic structure $\longleftrightarrow$ Space + constraints

An algebra in $\mathbb{ISP}(\mathbf{L})$ is represented via a

1. Stone space X, together with
2. a family of "at-most-binary constraints", i.e., for each subset $I \subseteq X$ of cardinality ≤ 2, a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$.

satisfying certain compatibility conditions.

Space X + constraints $(\mathbf{A}_I)_I$ $\rightsquigarrow$ Algebra of continuous functions $f\colon X \to \mathbf{L}$ (with $\mathbf{L}$ discrete) satisfying all constraints, i.e. such that, for all $I \subseteq X$ with $|I| \leq 2$, $f|_I \in \mathbf{A}_I$.

1. Stone duality for Boolean algebras (1936): $\mathbf{L} = \{0, 1\}$.

   Space: a Stone space.

   Constraints: "none", i.e. $\mathbf{A}_I := \{0, 1\}^I$.

   Boolean algebra $\cong$ algebra of continuous functions from a Stone space X to $\{0, 1\}$ with pointwise operations.

2. Priestley duality (1969): $\mathbf{L} = \{0, 1\}$.

   Space: a Stone space.

   Constraints: the order, i.e. $\mathbf{A}_I := \{\text{order-preserving } I \to \{0, 1\}\}$.

   Bounded distributive lattice $\cong$ algebra of order-preserving continuous functions from a Priestley space X to $\{0 < 1\}$.

"$I \subseteq_2 X$" means "I is a subset of X of cardinality at most 2".

Definition

A *Priestley* **L**-*space* consists of a Stone space X and, for each $I \subseteq_2 X$, of a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$ s.t.

1. (**Separation**) For all distinct $x, y \in X$, there is $f \in \mathbf{A}_{\{x,y\}}$ s.t. $f(x) \neq f(y)$.
2. (**Local-to-global extension**) For all $I \subseteq_2 X$, every $f \colon I \to \mathbf{L}$ in $\mathbf{A}_I$ has a continuous extension $g \colon X \to \mathbf{L}$ (with $\mathbf{L}$ discrete) that satisfies all constraints, i.e., s.t., for all $J \subseteq_2 X$, $g|_J \in \mathbf{A}_J$.

Separation	$\leftrightsquigarrow$	Anti-symmetry of the order.
Local-to-global extension	$\leftrightsquigarrow$	for $x \nleq y$ there is a clopen upset U s.t. $x \in U$, $y \notin U$.

The algebra associated to a Priestley **L**-space is

$$\{f\colon X \to \mathbf{L} \mid f \text{ is continuous and satisfies all constraints}\}.$$

$\mathbf{K} \coloneqq$ Kleene algebra $\{0, U, 1\}$.

Example of a $\mathbf{K}$-Priestley space:

- $\alpha\mathbb{N} \coloneqq$ one-point compactification of the discrete space $\mathbb{N}$.
- Set $\mathbf{A}_{\varnothing} \coloneqq \mathbf{K}^{\varnothing}$.
- For every $x \in \alpha\mathbb{N}$ set $\mathbf{A}_{\{x\}} \coloneqq \mathbf{K}^{\{x\}}$.
- For distinct $x, y \in \mathbb{N}$, set $\mathbf{A}_{\{x,y\}} \coloneqq \mathbf{K}^{\{x,y\}}$.
- For $x \in \mathbb{N}$ set

$$\mathbf{A}_{\{x,\infty\}} \coloneqq \{f \colon \{x, \infty\} \to \mathbf{K} \mid f(\infty) = U \text{ or } f \equiv 0 \text{ or } f \equiv 1\}.$$

Dual Kleene algebra: algebra of functions $f \colon \alpha\mathbb{N} \to \mathbf{K}$ such that $f \equiv 0$ or $f \equiv 1$ or ($f(\infty) = U$ and $f(x) = U$ for all but finitely many $x \in \mathbb{N}$).

Duality theorem

Hypotheses on the generating algebra **L**:

0. **L** is finite.
1. **L** has a lattice reduct.
2. **L** has two distinct constants and every subalgebra of **L** is simple.
3. For each subalgebra **A** of **L**, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from **A** to **L**.

Theorem

Suppose **L** *satisifes (0–3). The following categories are dually equivalent.*

- $\mathbb{ISP}(\mathbf{L})$ *(with homomorphisms as morphisms)*
- *the category of Priestley* **L***-spaces (and appropriate morphisms).*

Our generalisation

Our main result: we give an analagous duality, where the generating algebra $\mathbf{L}$ is allowed to be infinite.

Our motivation:

- representation of some *MV-algebras*, i.e. the algebras of Łukasiewicz many-valued logic: $\{0, 1\}$ is replaced by $[0, 1]$.
- representation of some *positive MV-algebras*, i.e. the negation-free version of MV-algebras.

Hypotheses on **L**:

0. ~~**L** is finite.~~
1. **L** has a lattice reduct, i.e. **L** has two terms $\vee$ and $\wedge$ that satisfy the axioms of lattices.
2. **L** has two distinct constants and every subalgebra of **L** is simple (i.e. the only congruences of a subalgebra **A** are the diagonal and $\mathbf{A} \times \mathbf{A}$).
3. For each subalgebra **A** of **L**, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from **A** to **L**.

Theorem (Main result)

Suppose $\mathbf{L}$ *satisifes (1–3). The following categories are dually equivalent:*

- *the category of algebras* $\mathbf{A} \in \mathbb{ISP}(\mathbf{L})$ *s.t. for each* $a \in \mathbf{A}$ *the set*

$$\{h(a) \colon h \colon \mathbf{A} \to \mathbf{L} \text{ homomorphism}\}$$

  is finite.
- *the category of Priestley* $\mathbf{L}$*-spaces.*

An equivalent characterization for the algebras: there is a set X and an embedding of $\mathbf{A}$ into the algebra of functions $X \to \mathbf{L}$ with finite image.

E.g.: any finite product of $\mathbf{L}$, and any subalgebra of it.

Thank you!