Natural dualities with an infinite dualizing object

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Stone duality

1936: Stone's representation theorem for Boolean algebras:

Every Boolean algebra $\langle B; \vee, \wedge, 0, 1, \neg \rangle$ is isomorphic to a field $\langle \mathcal{A}; \cup, \cap, \varnothing, X, X \setminus (-) \rangle$ of subsets of some set X.

Stone gave a uniform way to represent each Boolean algebra as a field of sets: for a Boolean algebra $\mathbf{B}$, Stone constructed a topological space $X_{\mathbf{B}}$, and

$$
\begin{aligned}
\mathbf{B} &\cong \{\text{clopen } (= \text{closed} + \text{open}) \text{ subsets of } X_{\mathbf{B}}\} \\
&\cong \{f \colon X_{\mathbf{B}} \to \{0,1\} \mid f \text{ is continuous}\}.
\end{aligned}
$$

This gives a 1:1 correspondence between Boolean algebras and Stone spaces (a.k.a. profinite spaces or Boolean spaces).

Stone space $:=$ compact Hausdorff spaces where clopens separate points.

E.g.: finite discrete spaces, one-point compactifications of discrete spaces.

Stone duality

In fact, Stone proved much more than a representation theorem:

$$\mathsf{Bool} \cong \mathsf{Stone}^{\mathrm{op}}.$$

Homomorphisms $A \to B$ of Boolean algebras correspond to continuous functions $X_B \to X_A$.

Some good things of Stone duality

Algebraic questions in Bool can be answered by translating them into (often simpler) questions in Stone.

1. While a coproduct $\mathbf{A} + \mathbf{B}$ of Boolean algebras is often difficult to describe, its dual, $X_{\mathbf{A}+\mathbf{B}}$, is simply the cartesian product $X_{\mathbf{A}} \times X_{\mathbf{B}}$.
2. The free Boolean algebra ($=$ Lindenbaum-Tarski algebra) on κ generators has a simple dual description: 2^{κ}.
3. Congruences on a Boolean algebra dually correspond to closed subspaces.
4. The duality is "logarithmic": a product $\mathbf{A} \times \mathbf{B}$ of Boolean algebras corresponds to a sum of the corresponding spaces: $X_{\mathbf{A}\times\mathbf{B}} \cong X_{\mathbf{A}} + X_{\mathbf{B}}$. For example, the dual of a Boolean algebra of $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$ elements is the discrete space of $6 = 1 + 1 + 1 + 1 + 1 + 1$ elements.

Priestley duality

[1969, Priestley]: the category of bounded distributive lattices $\langle B; \vee, \wedge, 0, 1\rangle$ is dually equivalent to the category of Priestley spaces.

$$\mathsf{BDL} \cong \mathsf{Priestley}^{\mathrm{op}}.$$

Priestley space := **Stone space** equipped with a **partial order** such that if $x \not\leq y$ then there is a clopen upset U such that $x \in U$ and $y \notin U$.

Every bounded distributive lattice is isomorphic to the lattice of clopen upsets of a Priestley space X. Equivalently, to the lattice of order-preserving continuous functions from X to $\{0, 1\}$.

New classes of algebras

Dualities for various other classes of algebras (usually stemming from nonclassical logic) were found: distributive lattices, semilattices with 1 (Hofmann-Mislove-Stralka duality), relative Stone algebras, de Morgan algebras, median algebras, Stone algebras, double Stone algebras, Kleene algebras.

Benefits similar to those for Stone duality apply.

These dualities follow a similar pattern: there is a “special” algebra $\mathbf{L}$ (for logics, $\mathbf{L}$ is the algebra of truth values) and a duality is given for

$$\mathbb{ISP}(\mathbf{L}).$$

Examples:

- Boolean algebras $= \mathbb{ISP}(\{0,1\})$ (by the Ultrafilter Lemma).
- Bounded distributive lattices $= \mathbb{ISP}(\{0,1\})$.

Davey & Werner (1983) showed that many of these follow from a general result: duality for $\mathbb{ISP}(\mathbf{L})$ where $\mathbf{L}$ is a finite algebra with a near unanimity term. (E.g.: $\mathbf{L}$ has a lattice reduct.)

Example: for $\mathbf{L} = \{0, 1\}$ we get Stone/Priestley dualities.

These dualities are called "natural dualities".

We present the restriction of this theory under some stronger hypotheses on **L**:

0. **L** is finite.
1. **L** has a majority term (e.g., **L** has a lattice reduct).
2. **L** is finitely subdirectly irreducible (e.g. **L** has two distinct constants and every subalgebra of **L** is simple).
3. For each subalgebra **A** of **L**, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from **A** to **L**.

For example, the Boolean algebra $\mathbf{L} = \{0, 1\}$.

Algebraic structure $\longleftrightarrow$ Space + constraints

An algebra in $\mathbb{ISP}(\mathbf{L})$ is represented via a

1. Stone space X, together with
2. a family of "at-most-binary constraints", i.e., for each subset $I \subseteq X$ of cardinality ≤ 2, a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$.

satisfying certain compatibility conditions.

Space X + constraints $(\mathbf{A}_I)_I$ $\rightsquigarrow$ Algebra of continuous functions $f \colon X \to \mathbf{L}$ (with $\mathbf{L}$ topologically discrete) satisfying all constraints, i.e. such that, for all $I \subseteq X$ with $|I| \leq 2$, $f|_I \in \mathbf{A}_I$.

1. Stone duality for Boolean algebras (1936): $\mathbf{L} = \{0,1\}$.

   Space: a Stone space.

   Constraints: "none", i.e. $\mathbf{A}_I := \{0,1\}^I$.

   Boolean algebra $\cong$ algebra of continuous functions from a Stone space X to $\{0,1\}$ with pointwise operations.

2. Priestley duality (1969): $\mathbf{L} = \{0,1\}$.

   Space: a Stone space.

   Constraints: the order, i.e. $\mathbf{A}_I := \{\text{order-preserving } I \to \{0,1\}\}$.

   Bounded distributive lattice $\cong$ algebra of order-preserving continuous functions from a Priestley space X to $\{0 < 1\}$.

"$I \subseteq_2 X$" means "I is a subset of X of cardinality at most 2".

Definition

A *Priestley* **L**-*space* consists of a Stone space X and, for each $I \subseteq_2 X$, of a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$ s.t.

1. (**Separation**) For all distinct $x, y \in X$, there is $f \in \mathbf{A}_{\{x,y\}}$ s.t. $f(x) \neq f(y)$.
2. (**Local-to-global extension**) For all $I \subseteq_2 X$, every $f \colon I \to \mathbf{L}$ in $\mathbf{A}_I$ has a continuous extension $g \colon X \to \mathbf{L}$ (with $\mathbf{L}$ discrete) that satisfies all constraints, i.e., s.t., for all $J \subseteq_2 X$, $g|_J \in \mathbf{A}_J$.

Separation	$\leftrightsquigarrow$	Anti-symmetry of the order.
Local-to-global extension	$\leftrightsquigarrow$	for $x \not\leq y$ there is a clopen upset U s.t. $x \in U$, $y \notin U$.

The algebra associated to a Priestley **L**-space is

$$\{f\colon X \to \mathbf{L} \mid f \text{ is continuous and satisfies all constraints}\}.$$

Duality theorem

Hypotheses on the generating algebra $\mathbf{L}$:

0. $\mathbf{L}$ is finite.
1. $\mathbf{L}$ has a majority term.
2. $\mathbf{L}$ is finitely subdirectly irreducible.
3. For each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$.

Theorem

Suppose $\mathbf{L}$ satisifes (0–3). The following categories are dually equivalent.

- $\mathbb{ISP}(\mathbf{L})$ *(with homomorphisms as morphisms)*
- *the category of Priestley $\mathbf{L}$-spaces (and appropriate morphisms).*

Our generalisation

Our main result: we give an analagous duality, where the generating algebra **L** is allowed to be infinite.

Our motivation:

- representation of some *MV-algebras*, i.e. the algebras of Łukasiewicz many-valued logic: $\{0, 1\}$ is replaced by $[0, 1]$.
- representation of some *positive MV-algebras*, i.e. the negation-free version of MV-algebras.

Hypotheses on **L**:

0. ~~**L** is finite.~~
1. **L** has a majority term.
2. **L** is finitely subdirectly irreducible.
3. For each subalgebra **A** of **L**, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from **A** to **L**.

Theorem (Main result)

Suppose **L** *satisifes (1–3). The following categories are dually equivalent:*

- *the category of algebras* $\mathbf{A} \in \mathbb{ISP}(\mathbf{L})$ *s.t. for each* $a \in \mathbf{A}$ *the set*

  $$\{h(a) \colon h \colon \mathbf{A} \to \mathbf{L} \textit{ homomorphism}\}$$

  is finite.
- *the category of Priestley* **L***-spaces.*

An equivalent characterization for the algebras: there is a set X and an embedding of **A** into the algebra of functions $X \to \mathbf{L}$ with finite image.

E.g.: any finite product of **L**, and any subalgebra of a finite product of **L**.

For **L** finite, this coincides with the theorem mentioned in the finite case.

Thank you!