Quantifier alternation depth in universal Boolean doctrines

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Introduction

Formulas of first-order logic can be stratified by their quantifier alternation depth = number of blocks of nested $\forall$ and $\exists$.

Q. altern. d. $= 0$ (q.-free):

- $R(x, y)$,
- $\neg R(x, y) \vee S(z)$,
- $S(z) \to T(x, y, z)$.

Q. alternation depth ≤ 1:

- quantifier-free,
- $\exists z \neg S(z)$,
- $\forall x\, R(x, y)$,
- $\forall x \forall y\, R(x, y)$,
- Boolean combinations of these.

Q. alternation depth ≤ 2:

- previous ones,
- $\forall x \exists y\, R(x, y)$
- $\forall x \forall y \neg \forall z\, T(x, y, z)$,
- $\exists x \forall y\, T(x, y, z)$,
- Boolean combination of these.

A formula has quantifier alternation depth...

- $\ldots = 0$ iff it is equivalent to a quantifier-free formula.
- $\ldots \leq 1$ iff it is equivalent to a Boolean combination of quantifications of formulas with quantifier alternation depth $= 0$.
- $\ldots \leq 2$ iff it is equivalent to a Boolean combination of quantifications of formulas with quantifier alternation depth ≤ 1.
- etc...

The same thing can be done modulo a first-order theory $\mathcal{T}$.

- A formula has quantifier alternation depth $= 0$ modulo $\mathcal{T}$ iff it is $\mathcal{T}$-equivalent to a quantifier-free formula.
- etc. . .

Given a first-order theory $\mathcal{T}$, can we reconstruct the quantifier alternation depth modulo $\mathcal{T}$ from the universal Boolean syntactic doctrine associated to $\mathcal{T}$?

Equivalently, can we recover the notion of being quantifier-free modulo $\mathcal{T}$?

If we restrict to the case where $\mathcal{T}$ is empty, probably yes. (Cf. Maietti and Trotta (2023) for existential doctrines.)

However, for a general theory $\mathcal{T}$, the answer is no.

Example:

$$\mathcal{L} = \{R \text{ (unary)}\}, \qquad \mathcal{T} = \emptyset.$$

$$\mathcal{L}' = \{R \text{ (unary)}, S \text{ (nullary)}\}, \qquad \mathcal{T}' = \{\forall x\, R(x) \leftrightarrow S\}.$$

The syntactic doctrines associated to $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic, but the formula $\forall x\, R(x)$ is quantifier-free in $\mathcal{T}'$ but not in $\mathcal{T}$.

We add more structure to the notion of a universal Boolean doctrine which encodes the quantifier-alternation depth.

This is in response to M. Gehrke's invitation "to make some nice mathematics or category theory out of the decoupage of formulas" given by the stratification on the quantifier depth.

Preliminaries on doctrines

Boolean doctrines

Let BA be the category of Boolean algebras and Boolean homomorphisms and Pos the category of partially ordered sets and order-preserving functions.

Definition

A ***Boolean doctrine over*** C is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, where C is a category with finite products.

$$\mathsf{C}^{\mathrm{op}} \xrightarrow{\mathbf{P}} \mathsf{BA}$$

$$\begin{array}{ccc} X & & \mathbf{P}(X) \\ \uparrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathbf{P}(f)} \\ X' & & \mathbf{P}(X') \end{array}$$

Examples

(1) (Syntactic doctrine) Fix a first-order theory $\mathcal{T}$ in a first-order language $\mathcal{L}$ (without equality).
Define the category Ctx of contexts:

- an object is a finite list $(x_1, \dots, x_n)$ of distinct variables;
- a morphism
$$(t_1(\vec{x}), \dots, t_m(\vec{x}))\colon (x_1, \dots, x_n) \to (y_1, \dots, y_m)$$
is an m-tuple of terms in the context $\vec{x} = (x_1, \dots, x_n)$.

The functor $\mathsf{LT}^{\mathcal{T}}\colon \mathsf{Ctx}^{\mathrm{op}} \to \mathsf{BA}$ is defined as follows:

$$\mathsf{Ctx}^{\mathrm{op}} \xrightarrow{\mathsf{LT}^{\mathcal{T}}} \mathsf{BA}$$

$$\begin{array}{ccc} \vec{y} & \mathsf{LT}^{\mathcal{T}}(\vec{y}) & \ni \alpha(\vec{y}) \vdash_{\mathcal{T}} \beta(\vec{y}) \\ {\scriptstyle \vec{t}(\vec{x})}\big\uparrow & \big\downarrow{\scriptstyle [\vec{t}(\vec{x})/\vec{y}]} & \big\downarrow \\ \vec{x} & \mathsf{LT}^{\mathcal{T}}(\vec{x}) & \ni \alpha(\vec{t}(\vec{x})/\vec{y}) \end{array}$$

Examples

(2) (Subset doctrine) The functor $\mathscr{P}\colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{BA}$ maps a set X to its power set $\mathscr{P}(X)$, and a function $f\colon X' \to X$ to the preimage function

$$\mathscr{P}(f) \coloneqq f^{-1}[-]\colon \mathscr{P}(X) \to \mathscr{P}(X').$$

Universal Boolean doctrines

Definition

A ***universal Boolean doctrine over*** C is a Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the following properties:

(1) (Universal) For all $X, Y \in \mathsf{C}$, the function $\mathbf{P}(\mathrm{pr}_1)\colon \mathbf{P}(X) \to \mathbf{P}(X \times Y)$ has a right adjoint $\forall_X^Y$ in Pos, i.e.

$$\alpha \leq \forall_X^Y \beta \text{ in } \mathbf{P}(X) \iff \mathbf{P}(\mathrm{pr}_1)(\alpha) \leq \beta \text{ in } \mathbf{P}(X \times Y)$$

(2) (Beck-Chevalley condition) For every morphism $f\colon X' \to X$ in C, the following diagram in Pos commutes.

$$\begin{tikzcd} X \\ X' \arrow[u, "f"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{P}(X \times Y) \arrow[r, "\forall_X^Y"] \arrow[d, "\mathbf{P}(f \times \mathrm{id}_Y)"'] & \mathbf{P}(X) \arrow[d, "\mathbf{P}(f)"] \\ \mathbf{P}(X' \times Y) \arrow[r, "\forall_{X'}^Y"'] & \mathbf{P}(X') \end{tikzcd}$$

Examples

(1) (Syntactic doctrine) The right adjoint to $\mathrm{LT}^{\mathcal{T}}(\mathrm{pr}_1)\colon \mathrm{LT}^{\mathcal{T}}(\vec{x}) \to \mathrm{LT}^{\mathcal{T}}(\langle \vec{x}; \vec{y} \rangle)$, which maps $\alpha(\vec{x})$ to itself, is

$$\forall y_1 \dots \forall y_m \colon \mathrm{LT}^{\mathcal{T}}(\langle \vec{x}; \vec{y} \rangle) \to \mathrm{LT}^{\mathcal{T}}(\vec{x}).$$

(2) (Subset doctrine) Given two sets X and Y, the right adjoint $\forall_X^Y$ of $\mathrm{pr}_1^{-1}\colon \mathscr{P}(X) \to \mathscr{P}(X \times Y)$ is

$$\begin{aligned} \forall_X^Y \colon \mathscr{P}(X \times Y) &\longrightarrow \mathscr{P}(X) \\ S &\longmapsto \{x \in X \mid \text{for all } y \in Y,\ (x, y) \in S\}. \end{aligned}$$

Morphisms

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ be Boolean doctrines. A ***Boolean doctrine morphism*** from $\mathbf{P}$ to $\mathbf{R}$ is a pair $(M, \mathfrak{m})$ where $M\colon \mathsf{C} \to \mathsf{D}$ is a functor that preserves finite products and $\mathfrak{m}\colon \mathbf{P} \longrightarrow \mathbf{R} \circ M^{\mathrm{op}}$ is a natural transformation.

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[rr, "M^{\mathrm{op}}"] \arrow[dr, "\mathbf{P}"'] & \arrow[d, phantom, "\xrightarrow{\mathfrak{m}}"] & \mathsf{D}^{\mathrm{op}} \arrow[dl, "\mathbf{R}"] \\
& \mathsf{BA} &
\end{tikzcd}$$

Morphisms

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ and $\mathbf{R}\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{BA}$ be universal Boolean doctrines. A ***universal Boolean doctrine morphism*** from $\mathbf{P}$ to $\mathbf{R}$ is a Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathbf{R}$ s.t., for all $X, Y \in \mathsf{C}$,

$$\begin{tikzcd}
\mathbf{P}(X \times Y) \arrow[r, "\mathfrak{m}_{X\times Y}"] \arrow[d, "\forall_X^Y"'] & \mathbf{R}(M(X) \times M(Y)) \arrow[d, "\forall_{M(X)}^{M(Y)}"] \\
\mathbf{P}(X) \arrow[r, "\mathfrak{m}_X"'] & \mathbf{R}(M(X))
\end{tikzcd}$$

Models

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A ***Boolean model of* P** is a Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$, where $\mathscr{P}$ is the subsets doctrine.

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine. A ***universal Boolean model of* P** is a universal Boolean doctrine morphism $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$.

Example

Let $\mathcal{T}$ be a first-order theory and let $\mathrm{LT}^{\mathcal{T}}$ the correspondent syntactic doctrine.
A universal Boolean model of $\mathrm{LT}^{\mathcal{T}}$ corresponds precisely to a model of the theory $\mathcal{T}$ in the classical sense.

Quantifier alternation depth for doctrines

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a universal Boolean doctrine. A ***quantifier-free fragment of* P** is a subfunctor $\mathbf{P}_0\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ of $\mathbf{P}$ that "generates $\mathbf{P}$".

"Generates $\mathbf{P}$" means that, for each $X \in \mathsf{C}$, $\mathbf{P}(X) = \bigcup_{n\in\mathbb{N}} \mathbf{P}_n(X)$, where

- $\mathbf{P}_0(X)$ is already defined;
- $\mathbf{P}_{n+1}(X)$ is the Boolean subalgebra of $\mathbf{P}(X)$ generated by the union of the images of $\mathbf{P}_n(X \times Y)$ under $\forall_X^Y\colon \mathbf{P}(X \times Y) \to \mathbf{P}(X)$, for Y ranging in C.

Definition

A ***quantifier-stratified universal Boolean doctrine over*** C is a sequence

$$\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}_2 \hookrightarrow \dots$$

of functors $\mathbf{P}_n \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$, where each $\mathbf{P}_n$ is a subfunctor of $\mathbf{P}_{n+1}$, s.t.

(1) (Universal) For all $\beta \in \mathbf{P}_n(X \times Y)$ there is $\forall^Y_{X,n}\beta \in \mathbf{P}_{n+1}(X)$ s.t. for all $\alpha \in \mathbf{P}_{n+1}(X)$

$$\alpha \leq \forall^Y_{X,n}\beta \quad \text{in } \mathbf{P}_{n+1}(X) \quad \iff \quad \mathbf{P}_{n+1}(\mathrm{pr}_1)(\alpha) \leq i_{X\times Y,n}(\beta) \quad \text{in } \mathbf{P}_{n+1}(X \times Y).$$

$$\begin{tikzcd}
\beta \arrow[r, phantom, "\in"] & \mathbf{P}_n(X \times Y) \arrow[r, hook, "i_{X\times Y,n}"] \arrow[dr, dashed, "\forall^Y_{X,n}"'] & \mathbf{P}_{n+1}(X \times Y) & \\
& & \mathbf{P}_{n+1}(X) \arrow[r, phantom, "\ni"] & \alpha
\end{tikzcd}$$

(2) (Beck-Chevalley) For every $f \colon X' \to X$ and $n \in \mathbb{N}$

$$\begin{tikzcd}
\mathbf{P}_n(X' \times Y) \arrow[r, "\forall^Y_{X',n}"] \arrow[d, "\mathbf{P}_n(f\times \mathrm{id}_Y)"'] & \mathbf{P}_{n+1}(X') \arrow[d, "\mathbf{P}_{n+1}(f)"] \\
\mathbf{P}_n(X \times Y) \arrow[r, "\forall^Y_{X,n}"'] & \mathbf{P}_{n+1}(X)
\end{tikzcd}$$

Definition (Continues)

(3) (Restriction of universal) For all $X, Y \in \mathsf{C}$ and $n \in \mathbb{N}$, $\forall^{Y}_{X,n+1}$ restricts to $\forall^{Y}_{X,n}$:

$$\begin{tikzcd}
\mathbf{P}_n(X \times Y) \arrow[r, hook, "i_{X\times Y,n}"] \arrow[dr, "\forall^{Y}_{X,n}"'] & \mathbf{P}_{n+1}(X \times Y) \arrow[dr, "\forall^{Y}_{X,n+1}"] & \\
& \mathbf{P}_{n+1}(X) \arrow[r, hook, "i_{X,n+1}"'] & \mathbf{P}_{n+2}(X)
\end{tikzcd}$$

(4) (Generation) For all $X \in \mathsf{C}$ and $n \in \mathbb{N}$, the Boolean algebra $\mathbf{P}_{n+1}(X)$ is generated by the union of the images of the functions $\forall^{Y}_{X,n} \colon \mathbf{P}_n(X \times Y) \to \mathbf{P}_{n+1}(X)$ for Y ranging in C.

Recap:

(1) Universal.

(2) Beck-Chevalley.

(3) Restriction of universal.

(4) Generation.

Proposition

There is a one-to-one correspondence between

(1) pairs $(\mathbf{P}_0, \mathbf{P})$ where $\mathbf{P}$ is a universal Boolean doctrine and $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}$.

(2) quantifier-stratified universal Boolean doctrines $\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}_2 \hookrightarrow \dots$

Given $(\mathbf{P}_0, \mathbf{P})$, one defines inductively $\mathbf{P}_n$.

Given $\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}_2 \hookrightarrow \dots$, one sets $\mathbf{P}$ as the colimit of $\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}_2 \hookrightarrow \dots$

Question

For each $n \in \mathbb{N}$, what is a characterization of the sequences of the form

$$\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \dots \mathbf{P}_{n-1} \hookrightarrow \mathbf{P}_n$$

for some quantifier-stratified universal Boolean doctrine $\mathbf{P}_0 \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}_2 \hookrightarrow \dots$?

We don't know yet.

We know for $n = 0$, i.e. the quantifier-free fragments: $\mathbf{P}_0$ is a Boolean doctrine.

- Immediate: any quantifier-free fragment of a universal Boolean doctrine is a Boolean doctrine.
- More difficult: the converse: we must show that any Boolean doctrine $\mathbf{P}_0$ is the quantifier-free fragment of some universal Boolean doctrine.

Quantifier completion of a Boolean doctrine

Existence of the quantifier completion

Theorem

Every Boolean doctrine $\mathbf{P}$ *over a small base category has a quantifier completion* $\mathbf{P}^{\forall}$*, i.e. a free universal Boolean doctrine wrt* $\mathbf{P}$*.*

$$\begin{tikzcd} \mathbf{P} \arrow[r] \arrow[dr] & \mathbf{P}^{\forall} \arrow[d, dashed] \\ & \mathbf{R} \end{tikzcd}$$

(We ignore 2-categorical aspects, for simplicity.)

Idea of the proof. Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small.

(1) The axioms of universal Boolean doctrines (over a fixed base category) are quasi-equations in a many-sorted language. $\rightsquigarrow$ There is a free many-sorted algebra $\mathbf{P}^{\forall} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the universal property over all Boolean doctrines $\mathbf{R}$ over C.

(2) $\mathbf{P}^{\forall} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ has the desired univ. property (over any base category). □

Characterization of quantifier-free fragments

Is $\mathbf{P}$ a quantifier-free fragment of $\mathbf{P}^{\forall}$?

Theorem

Let $\mathbf{P}$ *be a Boolean doctrine over a small base category. The quantifier completion morphism*

$$\mathbf{P} \to \mathbf{P}^{\forall}$$

is fiberwise injective.

Sketch of proof. We use a completeness theorem for Boolean doctrines saying that two distinct elements in a fiber of a Boolean doctrine are separated by a Boolean model.

$$\begin{tikzcd}
\mathbf{P} \arrow[r] \arrow[dr] & \mathbf{P}^{\forall} \arrow[d, dashed] \\
 & \mathscr{P}
\end{tikzcd}$$

□

Characterization of quantifier-free fragments

Corollary

Boolean doctrines are precisely the quantifier-free fragments of some universal Boolean doctrine.

Let $\mathbf{P}_0$ be a Boolean doctrine over C (small). Let $\mathbf{P}_0^\forall$ be its quantifier-completion and recall that $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}_0^\forall$.

For each $S \in \mathsf{C}$

$\mathbf{P}_1(S) :=$ Boolean subalgebra of $\mathbf{P}_0^\forall(S)$ generated by the union of the images of $\mathbf{P}_0(S \times Y)$ under $\forall_S^Y \colon \mathbf{P}_0^\forall(S \times Y) \to \mathbf{P}_0^\forall(S)$, $Y \in \mathsf{C}$.

$$
\begin{tikzcd}
\mathbf{P}_0(S \times Y) \arrow[r, hook] \arrow[dr, dashed] & \mathbf{P}_0^\forall(S \times Y) \arrow[dr, "\forall_S^Y"] & \\
& \mathbf{P}_1(S) \quad \subseteq & \mathbf{P}_0^\forall(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{P}_0 \arrow[rr, hook] \arrow[dr, hook, dashed] & & \mathbf{P}_0^\forall \\
& \mathbf{P}_1 \arrow[ur, hook, dashed] &
\end{tikzcd}
$$

This gives a subfunctor $\mathbf{P}_1$ of $\mathbf{P}_0^\forall$.

Intuitively, $\mathbf{P}_1$ freely adds one layer of quantification alternation to $\mathbf{P}_0$.

$$\begin{tikzcd}
\mathbf{P}_0 \arrow[rr, hook] \arrow[dr, hook, dashed] & & \mathbf{P}_0^{\forall} \\
& \mathbf{P}_1 \arrow[ur, hook, dashed] &
\end{tikzcd}$$

Goal: describe $\mathbf{P}_1$ explicitly.

Example, when is $(\forall x\, \alpha(x)) \wedge (\forall y\, \beta(y))$ below $(\forall z\, \gamma(z)) \vee (\forall w\, \delta(w))$?

Our approach: when every model of $\mathbf{P}_0$ satisfying $\forall x\, \alpha(x)$ and $\forall y\, \beta(y)$ also satisfies $\forall z\, \gamma(z)$ or $\forall w\, \delta(w)$.

To make it into an explicit condition about $\mathbf{P}_0$, α, β, γ and δ, we take a detour about models.

Characterization of classes of universally valid formulas

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, $(M, \mathfrak{m})\colon \mathbf{P} \to \mathscr{P}$ a Boolean model of $\mathbf{P}$. For each $X \in \mathsf{C}$, we define:

$$F_X^{(M,\mathfrak{m})} \coloneqq \{\alpha \in \mathbf{P}(X) \mid \text{for all } x \in M(X),\ x \in \mathfrak{m}_X(\alpha)\}.$$

Let $\{x_1, x_2, \dots\}$ be a countable set of variables, $\mathcal{L}$ a language, $\mathcal{T}$ a q.f. theory in $\mathcal{L}$, M a model for $\mathcal{T}$. For each $n \in \mathbb{N}$ we define:

$$F_n^M \coloneqq \{\alpha(x_1, \dots, x_n) \text{ quantifier-free} \mid M \vDash \forall x_1 \dots \forall x_n\, \alpha(x_1, \dots, x_n)\}.$$

Goal: characterize the families of the form $(F_X^{(M,\mathfrak{m})})_{X \in \mathsf{C}}$ for some model $(M, \mathfrak{m})$ (when the base category C is small).

Answer: these families are the ***universal ultrafilters***.

Universal ultrafilter

Definition

Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine. A ***universal ultrafilter for P*** is a family $(F_X)_{X\in\mathsf{C}}$, with $F_X \subseteq \mathbf{P}(X)$, s.t.:

- (Closure under reindexings) For all $f\colon X \to Y$ and $\alpha \in F_Y$, $\mathbf{P}(f)(\alpha) \in F_X$;
- (Filterness) For all $X \in \mathsf{C}$, F_X is a filter of $\mathbf{P}(X)$;
- (Binary primeness) For all $\alpha_1 \in \mathbf{P}(X_1)$ and $\alpha_2 \in \mathbf{P}(X_2)$, if $\mathbf{P}(\mathrm{pr}_1)(\alpha_1) \vee \mathbf{P}(\mathrm{pr}_2)(\alpha_2) \in F_{X_1\times X_2}$ then $\alpha_1 \in F_{X_1}$ or $\alpha_2 \in F_{X_2}$;
- (0-ary primeness) $\bot_{\mathbf{P}(\mathbf{t})} \notin F_{\mathbf{t}}$.

$$F^M_{\vec{x}} \coloneqq \{\alpha(\vec{x}) \text{ quantifier-free } \mid M \vDash \forall \vec{x}\, \alpha(\vec{x})\}.$$

Theorem

Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $F = (F_X)_{X \in \mathsf{C}}$ be a family with $F_X \subseteq \mathbf{P}(X)$ for each $X \in \mathsf{C}$. The following conditions are equivalent.

(1) *There is a Boolean model $(M, \mathfrak{m})$ of $\mathbf{P}$ such that, for every $X \in \mathsf{C}$,*

$$F_X = \{\alpha \in \mathbf{P}(X) \mid \textit{for all } x \in M(X),\, x \in \mathfrak{m}_X(\alpha)\}.$$

(2) *F is a universal ultrafilter for $\mathbf{P}$.*

Sketch of proof. (1) $\Rightarrow$ (2): Easy.

(2) $\Rightarrow$ (1): more difficult: given a universal ultrafilter, we should build a suitable model.

We mimic Henkin's proof of the Completeness Theorem. □

The model obtained from the universal ultrafilter is not canonical: we use the axiom of choice to extend **P** to a "rich" Boolean doctrine.

Free one-step construction

Back to the question: how do we freely add one layer of quantification to a Boolean doctrine $\mathbf{P}_0$?

$$
\begin{tikzcd}
\mathbf{P}_0 \arrow[rr, hook] \arrow[dr, hook, dashed] & & \mathbf{P}_0^{\forall} \\
& \mathbf{P}_1 \arrow[ur, hook, dashed] &
\end{tikzcd}
$$

We describe $\mathbf{P}_1$ in terms of $\mathbf{P}_0$, and then construct $\mathbf{P}_1$ via generators and relations.

We characterize the order in the fibers $\mathbf{P}_1(S)$ ($S \in \mathsf{C}$) in terms of $\mathbf{P}_0$.

Enough: to describe when a finite conjunction of universal quantifications is below a finite disjunction of universal quantifications.

$$(\forall x\, \alpha(x)) \wedge (\forall y\, \beta(y)) \leq (\forall z\, \gamma(z)) \vee (\forall w\, \delta(w)) \quad \text{in } \mathbf{P}_1$$

$\Downarrow$ univ. prop. quantif. completion

every model of $\mathbf{P}_0$ satisfying $\forall x\, \alpha(x)$ and $\forall y\, \beta(y)$ satisfies $\forall z\, \gamma(z)$ or $\forall w\, \delta(w)$

$\Updownarrow$ characterization univ. ultraf.

every universal ultrafilter of $\mathbf{P}_0$ containing $\alpha(x)$ and $\beta(y)$ contains $\gamma(z)$ or $\delta(w)$

$\Downarrow$ univ. ultraf. lemma

there are terms $t_1(z,w), \dots, t_n(z,w), s_1(z,w), \dots, s_m(z,w)$ s.t.

$$\bigwedge_{i=1}^{n} \alpha(t_i(z,w)) \wedge \bigwedge_{j=1}^{m} \beta(s_j(z,w)) \leq \gamma(z) \vee \delta(w) \quad \text{in } \mathbf{P}_0.$$

Theorem

Let $\mathbf{P}_0$ be a Boolean doctrine over C (small) and $\mathbf{P}_0^{\forall}$ its quantifier completion. Let $(\alpha_i \in \mathbf{P}_0(Y_i))_{i=1,\dots,\bar{i}}$, $(\beta_j \in \mathbf{P}_0(Z_j))_{j=1,\dots,\bar{j}}$. TFAE.

(1) *In $\mathbf{P}_0^{\forall}(\mathbf{t})$ (and thus, in $\mathbf{P}_1(\mathbf{t})$)*

$$\bigwedge_{i=1}^{\bar{i}} \forall_{\mathbf{t}}^{Y_i} \alpha_i \leq \bigvee_{j=1}^{\bar{j}} \forall_{\mathbf{t}}^{Z_j} \beta_j.$$

(2) *There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, and $(g_i \colon \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\dots,n}$ such that*

$$\bigwedge_{i=1}^{n} \mathbf{P}_0(g_i)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}_0(\mathrm{pr}_j)(\beta_j).$$

Meaning: let $\mathcal{T}$ be a q.f. theory, let $(\alpha_i(y_i))_{i=1,\dots,\bar{i}}$, $(\beta(z_j))_{j=1,\dots,\bar{j}}$ be q.f. formulas. TFAE.

(1) $\bigwedge_{i=1}^{\bar{i}} \forall y_i \, \alpha_i(y_i) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \forall z_j \, \beta_j(z_j)$.

(2) There are $n \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$ and terms $(g_i(\vec{z}))_{i \in \{1,\dots,n\}}$ such that

$$\bigwedge_{i=1}^{n} \alpha_{l_i}(g_i(\vec{z})) \vdash_{\mathcal{T}} \bigvee_{j=1}^{\bar{j}} \beta_j(z_j).$$

Example

If $\alpha(x)$ is a q.f. formula and β is a q.f. closed formula, when does $\forall x\, \alpha(x)$ imply β modulo a given q.f. theory $\mathcal{T}$? Precisely when there are $n \in \mathbb{N}$ and 0-ary terms $c_1, \ldots, c_n$ such that

$$\bigwedge_{i=1}^{n} \alpha(c_i) \vdash_{\mathcal{T}} \beta.$$

Corollary (Herbrand's theorem)

Let $\mathbf{P}_0$ be a Boolean doctrine with small base category, let $\mathbf{P}_0^\forall$ be its quantifier completion and let $\alpha \in \mathbf{P}_0(X)$. TFAE.

(1) In $\mathbf{P}_0^\forall(\mathbf{t})$ (and thus, in $\mathbf{P}_1(\mathbf{t})$)
$$\top_{\mathbf{P}_0^\forall(\mathbf{t})} \leq \exists_{\mathbf{t}}^X \alpha.$$

(2) There are $n \in \mathbb{N}$, and $(c_k \colon \mathbf{t} \to X)_{k=1,\dots,n}$ such that (in $\mathbf{P}_0(\mathbf{t})$)
$$\top_{\mathbf{P}_0(\mathbf{t})} \leq \bigvee_{k=1}^{n} \mathbf{P}_0(c_k)(\alpha).$$

Meaning: let $\mathcal{T}$ be a q.f. theory and $\alpha(x_1, \dots, x_v)$ a q.f. formula. The condition
$$\top \vdash_{\mathcal{T}} \exists x_1 \dots \exists x_v \, \alpha(x_1, \dots, x_v)$$
holds if and only if there are $n \in \mathbb{N}$ and lists of 0-ary terms $(c_k^1, \dots, c_k^v)_{k=1,\dots,n}$ such that
$$\top \vdash_{\mathcal{T}} \bigvee_{k=1}^{n} \alpha(c_k^1, \dots, c_k^v).$$

We construct $\mathbf{P}_1(\mathbf{t})$ from $\mathbf{P}_0$ via generators and relations.

Idea: the elements of $\mathbf{P}_1(\mathbf{t})$ are Boolean combinations of universal closures $\forall_{\mathbf{t}}^{Y}\alpha$ of formulas $\alpha \in \mathbf{P}_0(Y)$.

Some of these Boolean combinations may be identified.

We construct $\mathbf{P}_1(\mathbf{t})$ as a quotient $\mathrm{Free}_1^{\mathbf{P}_0}(\mathbf{t})$ of the free Boolean algebra over

$$\bigsqcup_{Y \in \mathsf{C}} \mathbf{P}_0(Y).$$

(Intuition: an element $\alpha \in \mathbf{P}_0(Y)$, when thought in $\mathrm{Free}_1^{\mathbf{P}_0}(\mathbf{t})$, denotes its universal closure $\forall_{\mathbf{t}}^{Y}\alpha$.)

Let B be the free Boolean algebra over $\bigsqcup_{Y \in \mathsf{C}} \mathbf{P}_0(Y)$

For $\alpha \in \mathbf{P}_0(Y)$, we write $\forall^Y_{\mathbf{t}} \alpha$ for the image of α under the free map $\bigsqcup_{Y \in \mathsf{C}} \mathbf{P}_0(Y) \to B$. Let $\sim$ be the Boolean congruence on B generated by the following relations: for each $n \in \mathbb{N}$, $l_1, \ldots, l_n \in \{1, \ldots, \bar{i}\}$ and $(g_i \colon \prod_{j=1}^{\bar{j}} Z_j \to Y_{l_i})_{i=1,\ldots,n}$ such that

$$\bigwedge_{i=1}^{n} \mathbf{P}_0(g_i)(\alpha_{l_i}) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}_0(\mathrm{pr}_j)(\beta_j).$$

we impose the relation

$$\Big[\bigwedge_{i=1}^{\bar{i}} \forall^{Y_i}_{\mathbf{t}} \alpha_i\Big] \leq \Big[\bigvee_{j=1}^{\bar{j}} \forall^{Z_j}_{\mathbf{t}} \beta_j\Big]$$

in $B/{\sim}$.

$$\mathrm{Free}_1^{\mathbf{P}_0}(\mathbf{t}) \coloneqq B/{\sim}.$$

With similar constructions, for each $S \in \mathsf{C}$ we define an appropriate quotient

$$\mathrm{Free}_1^{\mathbf{P}_0}(S)$$

of the free Boolean algebra over $\bigsqcup_{Y \in \mathsf{C}} \mathbf{P}_0(S \times Y)$.

Proposition

Let $\mathbf{P}_0 \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine, with C small. Let $\mathbf{P}_0^{\forall}$ be a quantifier completion of $\mathbf{P}_0$ and let $\mathbf{P}_1$ be defined from $\mathbf{P}_0$ and $\mathbf{P}_0^{\forall}$ as before. For every $S \in \mathsf{C}$, the Boolean algebras $\mathrm{Free}_1^{\mathbf{P}_0}(S)$ and $\mathbf{P}_1(S)$ are isomorphic.

We can extend the assignment $\mathrm{Free}_1^{\mathbf{P}_0}$ to morphisms of C, and get $\mathrm{Free}_1^{\mathbf{P}_0} \cong \mathbf{P}_1$.

Conclusion

The Boolean doctrine $\mathrm{Free}_1^{\mathbf{P}_0}$ (defined using generators and relations) freely adds a layer of quantifiers to $\mathbf{P}_0$.

Future work

Future work

- Let $(\mathbf{P}_n)_{n\in\mathbb{N}}$ be a quantifier-stratified universal Boolean doctrine. For each $n \in \mathbb{N}$, what are the properties satisfied by the tuple $(\mathbf{P}_0, \dots, \mathbf{P}_n)$? For now, we only addressed the case $n = 0$ (for a small base cateogry): $\mathbf{P}_0$ is a Boolean doctrine.

- How to freely construct $\mathbf{P}_{n+1}$ from $(\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_n)$?

- Provide a stepwise construction of the quantifier completion $\mathbf{P}^{\forall}$ of a Boolean doctrine $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ (with C small).

M. Abbadini, F. Guffanti. Quantifier alternation depth in universal Boolean doctrines. https://arxiv.org/abs/2404.08551.

Thank you for your attention!

Appendix: richness

Richness

Obtaining a model out of a universal ultrafilter is easy if we have *richness*.

Recall that a maximally consistent deductively closed first-order theory $\mathcal{T}$ is *rich* if for every formula $\exists x\, \beta(x) \in \mathcal{T}$ there is a 0-ary term c (a "witness") such that $\beta(c) \in \mathcal{T}$.

Definition

Let $(F_X)_{X \in \mathsf{C}}$ be a universal ultrafilter for a Boolean doctrine $\mathbf{P}$ over C. We say that $\mathbf{P}$ is ***rich with respect to*** $(F_X)_{X\in\mathsf{C}}$ if, for all $X \in \mathsf{C}$ and $\alpha \in \mathbf{P}(X) \setminus F_X$, there is $c \colon \mathbf{t} \to X$ such that $\mathbf{P}(c)(\alpha) \notin F_{\mathbf{t}}$.

Idea:

$$M \nvDash \forall x\, \alpha(x) \iff M \vDash \neg\forall x\, \alpha(x) \iff M \vDash \exists x\, \neg\alpha(x) \implies M \vDash \neg\alpha(c) \iff M \nvDash \alpha(c)$$

Universal ultrafilter + Richness $\rightsquigarrow$ Model

Proposition

Let $(F_X)_{X\in\mathsf{C}}$ be a universal ultrafilter for a Boolean doctrine $\mathbf{P}$ over C such that $\mathbf{P}$ is rich wrt $(F_X)_{X\in\mathsf{C}}$. There is a Boolean model $(M,\mathfrak{m})$ of $\mathbf{P}$ such that, for all $X\in\mathsf{C}$,

$$F_X = \{\alpha\in\mathbf{P}(X) \mid \text{for all } x\in M(X),\ x\in\mathfrak{m}_X(\alpha)\}.$$

Proof. Set $M := \mathrm{Hom}(\mathbf{t},-)\colon \mathsf{C}\to\mathsf{Set}$, and let $\mathfrak{m}\colon \mathbf{P}\to\mathscr{P}\circ M^{\mathrm{op}}$ be such that for $X\in\mathsf{C}$,

$$\begin{aligned}\mathfrak{m}_X\colon \mathbf{P}(X) &\longrightarrow \mathscr{P}(\mathrm{Hom}(\mathbf{t},X))\\ \alpha &\longmapsto \{c\colon\mathbf{t}\to X \mid \mathbf{P}(c)(\alpha)\in F_{\mathbf{t}}\}.\end{aligned}$$

□

How to obtain a model out of a universal ultrafilter F also in the non-rich case?

We will produce a rich theory out of the non-rich one, when C is small.

Recipe for richness

- Extend the language C adding new constants meant to witness the failure of universal closures of the formulas not belonging to the universal ultrafilter F.
- Interpret the formulas in F in the extended language C′, producing a new class of formulas G. But G might fail to be a universal ultrafilter in C′ (because of the new formulas).
- Use an appropriate generalization of the ultrafilter lemma ("universal ultrafilter lemma") to extend G to a universal ultrafilter F' in the extended language.
- We might lack some witnesses for the new formulas (involving the new constants) not belonging to F'.

The rich theory is obtained as a colimit after ω steps.

$$
\begin{array}{ccccccccc}
\mathsf{C} & \rightsquigarrow & \mathsf{C}' & = & \mathsf{C} \cup \{c_\sigma\}_{X \in \mathsf{C}, \sigma \notin F_X} & & & & \\
 & & & & & & & & \\
F & \rightsquigarrow & G & \subseteq & F' & \not\ni & \sigma'??? \\
\rotatebox{-90}{$\ni$} & & \rotatebox{-90}{$\ni$} & & \rotatebox{-90}{$\ni$} & & \\
\sigma & & \sigma(c_\sigma/X) & & \sigma(c_\sigma/X) & &
\end{array}
$$