A duality for metrically complete lattice-ordered groups

Marco Abbadini

School of Computer Science, University of Birmingham, UK

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Representation theorem for Boolean algebras

Theorem (Stone, 1936)

Every Boolean algebra embeds into the power set $\mathcal{P}(X)$ of some set X.

Stone duality

Stone proved much more than a representation theorem: in modern terms,

Theorem (Stone, 1936)

The category of Boolean algebras and homomorphisms is dually equivalent to the category of Stone spaces and continuous functions.

Having a duality allows us to easily transfer problems from one side of the duality to the other.

For example, to count the number of quotients of a Boolean algebra, one can count the number of closed subspaces of the dual Stone space.

From Boolean algebras to MV-algebras

Classical propositional logic	Łukasiewicz propositional logic
Boolean algebras	MV-algebras
$\{0,1\}$	$[0,1]$

Dualities for MV-algebras

Some pleasant features of Stone duality for Boolean algebras are not available for the whole class of MV-algebras.

E.g.: while every Boolean algebra is an algebra of $\{0, 1\}$-valued functions (Stone, 1936), not every MV-algebra is an algebra of $[0, 1]$-valued functions.

Thus, one looks for dualities for subclasses of MV-algebras.

From MV-algebras to lattice-ordered groups

We give a duality for a certain class of MV-algebras.

We do so by giving a duality for a certain class of *unital Abelian lattice-ordered groups*.

Theorem (Mundici, 1986)

The category of MV-algebras is equivalent to the category of unital Abelian ℓ-groups.

$$\text{MV-algebras} \longleftrightarrow \text{unital Abelian lattice-ordered groups}$$

$$[0,1] \longleftrightarrow \mathbb{R}.$$

Abelian lattice-ordered groups

Definition

An *Abelian lattice-ordered group* (*Abelian ℓ-group*, for short) is an abelian group with a lattice order s.t.

Translation invariance: $x \leq y$ implies $x + z \leq y + z$.

Example

- $\mathbb{R}$, $\mathbb{Q}$, $\frac{1}{n}\mathbb{Z}$ ($n \in \mathbb{N}_{>0}$).
- Given a topological space X,

$$C(X, \mathbb{R}) \coloneqq \{f \colon X \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Unital Abelian lattice-ordered groups

If X is a compact space, in the Abelian ℓ-group $C(X, \mathbb{R})$ the constant function 1 has a special property: every element is below a multiple of 1.

Definition

A *strong order unit* of an Abelian ℓ-group $\mathbf{G}$ is an element $1 \in \mathbf{G}$ such that for any x there is $n \in \mathbb{N}$ such that

$$x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Definition

A *unital Abelian ℓ-group* is an Abelian ℓ-group with a designated strong order unit.

Examples of unital Abelian ℓ-groups

Example

Given a compact Hausdorff space X,

$$C(X,\mathbb{R}) \coloneqq \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\},$$

$$C(X,\mathbb{Q}) \coloneqq \{f\colon X \to \mathbb{Q} \mid f \text{ is continuous}\},$$

$$C\left(X,\frac{1}{n}\mathbb{Z}\right) \coloneqq \left\{f\colon X \to \frac{1}{n}\mathbb{Z} \mid f \text{ is continuous}\right\},$$

are unital Abelian ℓ-group.

We can also mix the values that the functions can take at different points...

Representation theorem

Goodearl & Handelman characterized the unital Abel. ℓ-groups arising as:

Let X be a compact Hausdorff space. For each $x \in X$, let $A_x \in \{\mathbb{R}\} \cup \{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\}$. The following is a unital Abelian ℓ-group:

$$\{f\colon X \to \mathbb{R} \mid f \text{ continuous and, for all } x \in X,\ f(x) \in A_x\}.$$

Example

- If $A_x = \mathbb{R}$ for all $x \in X$, then we get $C(X, \mathbb{R})$.
- If $A_x = \frac{1}{2}\mathbb{Z}$ for all $x \in X$, then we get $C(X, \frac{1}{2}\mathbb{Z})$.

Theorem (Goodearl and Handelman, 1980)

The unital Abelian ℓ-groups arising in this way are precisely the **metrically complete** *ones.*

Metrically complete unital Abelian ℓ-groups

In $C(X,\mathbb{R})$, $C(X,\frac{1}{n}\mathbb{Z})$, ..., the sup metric is complete.

$$\begin{aligned} d_\infty(f,g) &= \inf\left\{\lambda \in \mathbb{R}^+ \mid -\lambda \le f-g \le \lambda\right\} \\ &= \inf\left\{\frac{p}{q} \in \mathbb{Q}^+ \mid -\frac{p}{q} \le f-g \le \frac{p}{q}\right\} \\ &= \inf\left\{\frac{p}{q} \in \mathbb{Q}^+ \mid -p \le q(f-g) \le p\right\}. \end{aligned}$$

Definition

A unital Abelian ℓ-group $\mathbf{G}$ is *metrically complete* if

$$d(v,w) \coloneqq \inf\left\{\frac{p}{q} \in \mathbb{Q}^+ \mid -p1 \le q(v-w) \le p1\right\}.$$

defines a complete metric $d\colon \mathbf{G}\times\mathbf{G} \to \mathbb{R}$.

Non-example: $\mathbb{Q}$.

Representation theorem

The result by Goodearl & Handelman is a representation theorem for metrically complete unital Abelian ℓ-groups.

They can be represented as

$$\{f\colon X \to \mathbb{R} \mid f \text{ continuous and, for all } x \in X,\ f(x) \in A_x\}$$

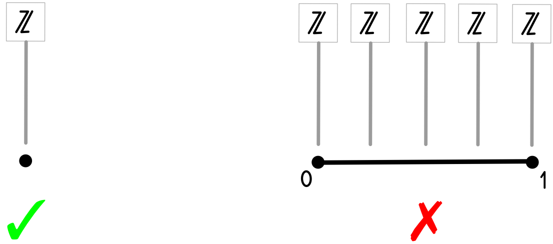
for some compact Hausdorff space X and some family $A_x \in \{\mathbb{R}\} \cup \{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\}$ (for $x \in X$).

Our aim

Our aim: make the Goodearl-Handelman representation into a **categorical duality**, so that we can transfer all problems that can be expressed in the categorical language.

Not a 1:1 correspondence

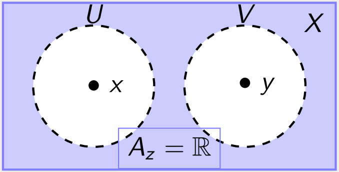
Goodearl & Handelman's representation is not a 1:1 correspondence.



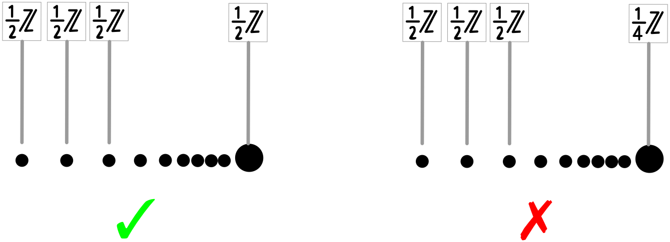
In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

Remedy: Separation

For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $A_z = \mathbb{R}$.



Not a 1:1 correspondence



In both cases

$$\left\{f\colon X \to \mathbb{R} \mid f \text{ cont. and, for all } x \in X,\, f(x) \in A_x\right\} =$$

$$= \left\{f\colon X \to \frac{1}{2}\mathbb{Z} \mid f \text{ is eventually constant}\right\}.$$

Remedy: Semicontinuity

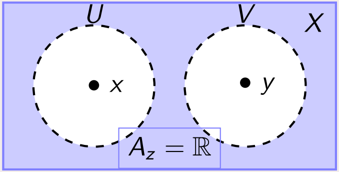
For every $n \in \mathbb{N}_{>0}$, $\{x \in X \mid A_x \subseteq \frac{1}{n}\mathbb{Z}\}$ is closed.

Normal a-spaces

Definition

A *normal a-space* (for "arithmetically normal arithmetic space") is a compact Hausdorff space X equipped with a family $(A_x)_{x \in X}$ with $A_x \in \{\mathbb{R}\} \cup \left\{ \frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0} \right\}$ such that

1. (Separation) For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $A_z = \mathbb{R}$.



2. (Semicontinuity) For every $n \in \mathbb{N}_{>0}$, $\left\{ x \in X \mid A_x \subseteq \frac{1}{n}\mathbb{Z} \right\}$ is closed.

Main result

Main result

The category of **metrically complete unital Abelian ℓ-groups** (and homomorphisms) is dually equivalent to the category of **normal a-spaces** (and continuous "denominator-decreasing" maps).

Metric. compl. unital Ab. ℓ-groups $\leftrightarrow$ *"metrically complete" MV-algebras*.

Corollary

The category of **normal a-spaces** *is dually equivalent to the category of* **metrically complete MV-algebras**.

Thank you!

M. Abbadini, V. Marra, L. Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. Preprint at arXiv:2210.15341.