Vietoris endofunctor for closed relations and its de Vries dual

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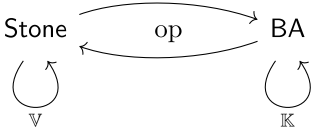
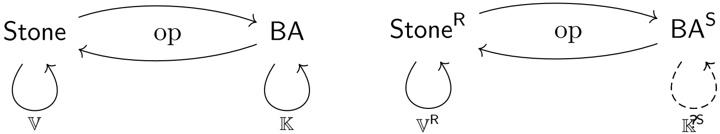
Kripke semantics connects modal logic with the Vietoris endofunctor.



Algebras for $\mathbb{K}$ = modal algebras.

Coalgebras for $\mathbb{V}$ = descriptive frames.

This is the coalgebraic approach to modal logic.



1. Stone duality has a natural extension to closed relations [Celani, 2018].

2. Vietoris has a natural extension to closed relations [Goy, Petrișan, Aiguier, 2021]! Which arised in studies of the usual Vietoris on functions.

3. We provide a description of the dual of $\mathbb{V}^{\mathsf{R}}$.

Natural notions $\rightarrow$ hope for applications! Our application: a resolution of an open problem on de Vries duality [Bezhanishvili, Bezhanishvili, Harding, 2015], concerning the usual Vietoris on compact Hausdorff spaces and continuous functions.

$$\mathsf{Stone} \underset{\longleftarrow}{\overset{\longrightarrow}{\text{op}}} \mathsf{BA}$$

Extended by Celani (2018) (see also Kurz, Moshier, Jung, 2023):

$$\mathsf{Stone}^{\mathsf{R}} \underset{\longleftarrow}{\overset{\longrightarrow}{\text{op}}} \mathsf{BA}^{\mathsf{S}}$$

closed relations subordinations

Closed relation $R\colon X \looparrowright Y$ is a subset $R \subseteq X \times Y$ that is closed; equivalently, such that

- R[closed] is closed,
- R^{-1}[closed] is closed.

Why we got into closed relations:

Every compact Hausdorff space X is a continuous image of a Stone space (e.g., its Gleason cover). So it can be presented via

$$\text{Stone space} + \text{closed equivalence relation.}$$

$$\begin{array}{ccccccc}
\mathsf{Stone} & \ni & \mathcal{G}(X) & \hookrightarrow & \mathcal{G}(Y) & \in & \mathsf{Stone} \\
 & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\
\mathsf{KHaus} & \ni & X & \xrightarrow{\ f\ } & Y & \in & \mathsf{KHaus}
\end{array}$$

Dual of a closed relation:

$$\begin{array}{c} X \\ \Big\downarrow R \\ Y \end{array} \qquad\qquad \begin{array}{c} \mathrm{Clop}(X) \\ \Big\uparrow S \\ \mathrm{Clop}(Y) \end{array}$$

For $V \in \mathrm{Clop}(Y)$ and $U \in \mathrm{Clop}(X)$:

$$V \, S \, U \iff R^{-1}[V] \subseteq U$$

Example:

$$\begin{array}{c} X \\ \Big\downarrow = \\ X \end{array} \qquad\qquad \begin{array}{c} \mathrm{Clop}(X) \\ \Big\uparrow \subseteq \\ \mathrm{Clop}(X) \end{array}$$

Subordination := a relation $S\colon A \nrightarrow B$ such that

$$\left(\bigvee_{i=1}^{n} a_i\right) S \left(\bigwedge_{j=1}^{m} b_j\right) \iff \forall i,j\ a_i\ S\ b_j.$$

Theorem (Celani, 2018)

$\mathsf{Stone}^{\mathsf{R}}$ *(closed relations) is dual to* BA^{S} *(subordinations).*

Definition (Vietoris hyperspace)

The Vietoris hyperspace $\mathbb{V}(X)$ of a Stone space X is the set of closed subsets of X, equipped with the topology generated by the following subsets of $\mathbb{V}(X)$, for U clopen of X:

$$\Diamond U \coloneqq \{K \in \mathbb{V}(X) \mid K \cap U \neq \varnothing\},$$
$$\Box U \coloneqq \{K \in \mathbb{V}(X) \mid K \subseteq U\}.$$

Vietoris functor on Stone:

$$\mathsf{Stone} \xrightarrow{\mathbb{V}} \mathsf{Stone}$$

$$\begin{array}{ccc} X & & \mathbb{V}(X) \\ \downarrow f & & \downarrow f[-] \\ Y & & \mathbb{V}(Y) \end{array}$$

Extension of $\mathbb{V}$ from Stone to $\mathsf{Stone}^{\mathsf{R}}$ [Goy, Petrișan, Aiguier, 2021]:

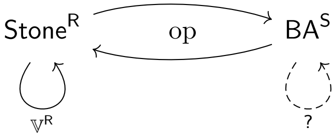
$$\mathsf{Stone}^{\mathsf{R}} \xrightarrow{\mathbb{V}^{\mathsf{R}}} \mathsf{Stone}^{\mathsf{R}}$$

$$\begin{array}{c} X \\ \Big\downarrow R \\ Y \end{array} \qquad \begin{array}{c} \mathbb{V}(X) \\ \Big\downarrow \mathbb{V}^{\mathsf{R}}(R) \\ \mathbb{V}(Y) \end{array}$$

(Egli-Milner:) For $K \in \mathbb{V}(X)$ and $L \in \mathbb{V}(Y)$,

$$K \; \mathbb{V}^{\mathsf{R}}(R) \; L \iff \begin{cases} \forall x \in K \; \exists y \in L : \; x \, R \, y, \\ \forall y \in L \; \exists x \in K : \; x \, R \, y. \end{cases}$$

It restricts to the usual Vietoris functor on continuous functions.



What is the dual of $\mathbb{V}^{\mathsf{R}}$?

$$\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$$

On objects: the same as for the dual of $\mathbb{V}$ on Stone (Abramsky, Johnstone, Kupke, Kurz, Venema, Vosmaer):

$$\frac{X \qquad \mathbb{V}(X)}{A \qquad \mathbb{K}(A)} \quad \text{Stone duality}$$

$$\mathbb{K}(A) = \frac{\mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \text{ preserves finite meets}, \Diamond = \neg\Box_\neg\}}$$

On morphisms:

$$\begin{array}{ccc} X & & \mathbb{V}(X) \\ \big\downarrow R & & \big\downarrow \mathbb{V}^{\mathsf{R}}(R) \\ Y & & \mathbb{V}(Y) \end{array}$$

$$\begin{array}{ccc} B & & \mathbb{K}(B) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_b, \Diamond_b \mid b \in B\})/{\sim} \\ \big\uparrow S & & \big\uparrow \mathbb{K}^{\mathsf{S}}(S)? \\ A & & \mathbb{K}(A) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})/{\sim} \end{array}$$

We shall describe when an element α of $\mathbb{K}(A)$ is $\mathbb{K}^{\mathsf{S}}(S)$-related with an element β of $\mathbb{K}(B)$.

Proposition

Given a Boolean algebra A. Every $\gamma \in \mathbb{K}(A)$ is (effectively) equal to

- (DNF) a finite join of elements of the form

$$\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b$$

  with each $a_i \leq b$;

- (CNF) a finite meet of elements of the form

$$\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}$$

  with each $c \leq d_j$.

$$\begin{array}{ccc} A & & \mathbb{K}(A) \\ \Big\downarrow S & & \Big\downarrow \mathbb{K}^{\mathsf{S}}(S)? \\ B & & \mathbb{K}(B) \end{array}$$

Enough to describe when

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b)\ \mathbb{K}^{\mathsf{S}}(S)\ (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

with $a_i \leq b$ and $c \leq d_j$.

$$\begin{array}{ccc} A & & \mathbb{K}(A) \\ \big\downarrow \leq & & \big\downarrow \leq \\ A & & \mathbb{K}(A) \end{array}$$

(With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \leq (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

Key idea: **$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$**. [Cederquist, Coquand, 1998]

E.g.: if A, B, C, D are clopens of a Stone space X with $A \subseteq C$ and $B \subseteq D$, then

$$\Diamond A \cap \Box B \subseteq \Diamond C \cup \Box D \iff A \subseteq C \text{ or } B \subseteq D.$$

$$\begin{array}{ccc} A & & \mathbb{K}(A) \\ \big\downarrow S & & \big\downarrow \mathbb{K}^{\mathsf{S}}(S) \\ B & & \mathbb{K}(B) \end{array}$$

(With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \; \mathbb{K}^{\mathsf{S}}(S) \; (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \; S \; c) \text{ or } (\exists j : b \; S \; d_j).$$

Key idea: **$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$**.

Theorem (A., Bezhanishvili, Carai, 2024)

The dual of the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ *is the following endofunctor* $\mathbb{K}^{\mathsf{S}} \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$:

- *On objects: it maps* A *to*

$$\mathbb{K}(A) := \frac{\mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \textit{ preserves finite meets}, \ \Diamond = \neg \Box_\neg\}}$$

- *On morphisms: it maps a subordination* $S \colon A \nrightarrow B$ *to the unique subordination* $\mathbb{K}^{\mathsf{S}}(S) \colon \mathbb{K}(A) \nrightarrow \mathbb{K}(B)$ *satisfying "*$\Diamond$*-with-*$\Diamond$ *or* $\Box$*-with-*$\Box$*".*

"$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$": (With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \ \mathbb{K}^{\mathsf{S}}(S) \ (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \ S \ c) \text{ or } (\exists j : b \ S \ d_j).$$

An application

De Vries duality is a duality for compact Hausdorff spaces, which associates to a compact Hausdorff space X the Boolean algebra of regular opens, together with the binary relation $\prec$ of well-insideness: $U \prec V \iff \mathrm{cl}(U) \subseteq V$.

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on compact Hausdorff spaces?

We piggyback on the duality between $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ and $\mathbb{K}^{\mathsf{S}} \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$.

Theorem (A., Bezhanishvili, Carai, 2024)

The de Vries dual of the Vietoris endofunctor on KHaus *is obtained by applying* $\mathbb{K}^{\mathsf{S}}$ *(= the dual of* $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$*), followed by a(n appropriate) MacNeille completion.*

$$X \xrightarrow{\;R\;} Y \qquad\qquad (B, \prec_B) \xleftarrow{\;S\;} (A, \prec_A)$$

$$\begin{array}{ccc} & (\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) \xleftarrow{\mathbb{K}^{\mathsf{S}}(S)} (\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \\ & \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \mathbb{V}(X) \xrightarrow{\mathbb{V}^{\mathsf{R}}(R)} \mathbb{V}(Y) & \mathbf{M}(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) \xleftarrow{\mathbf{M}(\mathbb{K}^{\mathsf{S}}(S))} \mathbf{M}(\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \end{array}$$

where $\mathbf{M}$ is an appropriate MacNeille completion functor.

Conclusions

Key ideas

1. Beyond functions; closed relations between Stone spaces ($\leftrightarrow$ subordinations between Boolean algebras).
   Especially: in dualities between "KHaus"-like and "lattice+proximity"-like structures.
   (Scott, Vickers, Jung, Sünderhauf, Moshier, Kegelmann, Kurz, ...)
2. "$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$" (Cederquist, Coquand):

$$\left(\bigwedge_i \Diamond_{a_i}\right) \wedge \Box_b \leq \Diamond_c \vee \left(\bigvee_j \Box_{d_j}\right) \Leftrightarrow (\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

3. Our packaging of these ideas:
   - Stone dual description of $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$;
   - de Vries dual description of $\mathbb{V} \colon \mathsf{KHaus} \to \mathsf{KHaus}$ and for relations.

M. Abbadini, G. Bezhanishvili, L. Carai.
Vietoris endofunctor for closed relations and its de Vries dual.
Topology Proceedings, to appear. Available on arxiv:2308.16823.