Vietoris endofunctor for closed relations and its de Vries dual

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Based on a homonymous paper with G. Bezhanishvili and L. Carai
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Algebras of propositional modal logic: Modal algebras

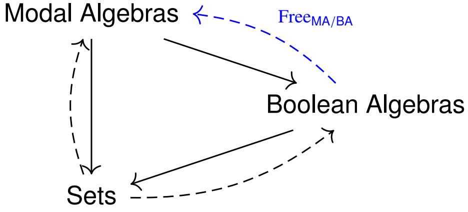
= Boolean algebra with unary operations $\Box$ and $\Diamond$ such that

$\Box = \neg\Diamond\neg$, $\Box$ pres. finite meets, $\Diamond$ pres. finite joins.

Free modal algebra over a set X = algebra of equivalence classes of modal propositional formulas with variables in X.

Free modal algebras are quite complex: already the free modal algebra over the empty set is not trivial:

$$0, 1, \Diamond 1, \Box 0, \Box\Diamond 1, \dots$$



To understand the free modal algebra $\mathrm{Free}_{\mathsf{MA/BA}}(A)$: *step-by-step method* ([Abramsky, 1988], [Ghilardi, 1995], [N. Bezhanishvili, Kurz, 2007]):

In general, $\mathrm{Free}_{\mathsf{MA/BA}}(A)$ has many elements of different shapes: for $a, b \in A$

$$a,\ b,\ \Box b,\ a \vee \Box b,\ \Diamond(a \vee \Box b), \dots$$

However, it is enough to understand the Boolean subalgebra of elements of pure rank 1:

$\Diamond a$, $\Box a$ ($a \in A$), and Boolean combinations of these.

I.e.: the elements of $\mathrm{Free}_{\mathsf{MA/BA}}(A)$ such that every element of A is under the scope of exactly one modality.

The subalgebra of elements of pure rank 1 is isomorphic to

$$\mathbb{K}(A) := \frac{\text{Free Boolean algebra over } \{\Diamond a \mid a \in A\} \sqcup \{\Box a \mid a \in A\}}{\Box = \neg\Diamond\neg,\ \Box \text{ pres. finite meets, } \Diamond \text{ pres. finite joins}}$$

This is easy to understand: there is an algorithm to compute when one element is less than or equal to another one.

Under side condition $a_i \leq b$ and $c \leq d_j$ ("**diamond-in-box**"):

$$(\Diamond a_1 \wedge \cdots \wedge \Diamond a_n \wedge \Box b) \leq (\Diamond c \vee \Box d_1 \vee \cdots \vee \Box d_m)$$

$$\Updownarrow$$

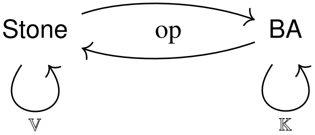
$$(\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

This is the "**$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$**" formula [Cederquist, Coquand, 1998].

$\mathbb{K}$ can be extended to an endofunctor on the category BA of Boolean algebras and Boolean homomorphisms.

$\mathrm{Free}_{\mathsf{MA/BA}}$ can be constructed "step-by-step" in terms of $\mathbb{K}$.

Under Stone duality, $\mathbb{K}$ corresponds to the Vietoris endofunctor $\mathbb{V}$ on the category of Stone spaces and continuous functions.



But actually Boolean homomorphisms are not the only interesting morphisms between Boolean algebras.

Definition

A ***subordination*** from a Boolean algebra A to a Boolean algebra B is a relation $S \subseteq A \times B$ s.t.

$$\left(\bigvee_{i=1}^{n} a_i \right) S \left(\bigwedge_{j=1}^{m} b_j \right) \Leftrightarrow \forall i, j \; (a_i \; S \; b_j).$$

$\approx$ approximable mappings in domain theory (Scott, 1982)

$\approx$ proximities / contact relations in the region-based theories of spaces.

[Jung & Sünderhauf, 1995], [Kegelmann, 1999], [Moshier, 2004], [Celani, 2018], [Kurz, Moshier, Jung, 2023]:

Subordinations between Boolean algebras are dual to closed relations between Stone spaces.

A *closed relation* from a Stone space X to a Stone space Y is a closed subset of $X \times Y$.

Our key observation: given a subordination S from A to B, a simple generalization of the formula "$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$" gives a subordination $\mathcal{S}$ from $\mathbb{K}(A)$ to $\mathbb{K}(B)$!

Under side condition: $a_i \leq b$ and $c \leq d_j$ ("diamond-in-box"):

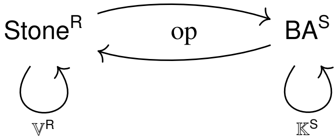
$$(\Diamond a_1 \wedge \cdots \wedge \Diamond a_n \wedge \Box b) \, \mathcal{S} \, (\Diamond c \vee \Box d_1 \vee \cdots \vee \Box d_m)$$

$$\Updownarrow$$

$$(\exists i : a_i \; S \; c) \text{ or } (\exists j : b \; S \; d_j).$$

Contributions

1. We turn the assignment $\mathbb{K}$ (capturing modal formulas of pure rank 1) into an endofunctor $\mathbb{K}^{\mathsf{S}}$ on the category BA^{S} of Boolean algebras and *subordinations*.
2. We prove that $\mathbb{K}^{\mathsf{S}}$ is the *Stone dual* of the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}}$ (introduced in [Goy, Petrişan, Aiguier, 2021]) on the category $\mathsf{Stone}^{\mathsf{R}}$ of Stone spaces and closed relations.



3. We answer [G. & N. Bezhanishvili, Harding, 2015]: using $\mathbb{K}^{\mathsf{S}}$, we describe the endofunctor on de Vries algebras that is dual to the Vietoris endofunctor on compact Hausdorff spaces.

M. Abbadini, G. Bezhanishvili, L. Carai. Vietoris endofunctor for closed relations and its de Vries dual. *Topology Proceedings*. Openly accessible preprint on arXiv.

Appendix

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on the category of compact Hausdorff spaces and continuous functions?

De Vries duality connects compact Hausdorff spaces with (Stone spaces and) Boolean algebras.

It associates to a compact Hausdorff space X the Boolean algebra of regular open sets together with the binary relation defined by $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.

Every compact Hausdorff space X is a continuous image of a Stone space (e.g., its Gleason cover). So it can be presented via

Stone space + closed equivalence relation.

$$
\begin{array}{ccccccc}
\text{Stone} & \ni & \mathcal{G}(X) & \hookrightarrow & \mathcal{G}(Y) & \in & \text{Stone} \\
 & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\
\text{KHaus} & \ni & X & \xrightarrow{f} & Y & \in & \text{KHaus}
\end{array}
$$

Theorem (A., Bezhanishvili, Carai, 2024)

The de Vries dual of the Vietoris endofunctor on KHaus *is obtained by applying* $\mathbb{K}^{\mathsf{S}}$ *(= the dual of* $\mathbb{V}^{\mathsf{R}}\colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$*), followed by a(n appropriate) MacNeille completion.*

$$X \xrightarrow{\;R\;} Y \qquad\qquad (B, \prec_B) \xleftarrow{\;S\;} (A, \prec_A)$$

$$\begin{array}{ccc}
(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) & \xleftarrow{\;\mathbb{K}^{\mathsf{S}}(S)\;} & (\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \\
\downarrow & & \downarrow \\
\mathbf{M}(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) & \xleftarrow{\;\mathbf{M}(\mathbb{K}^{\mathsf{S}}(S))\;} & \mathbf{M}(\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A))
\end{array}$$

$$\mathbb{V}(X) \xrightarrow{\;\mathbb{V}^{\mathsf{R}}(R)\;} \mathbb{V}(Y)$$

where **M** is an appropriate MacNeille completion functor.