Quantifier-free fragments and quantifier alternation depth in doctrines

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Algebraic logic

Algebraic logic: identifies logically equivalent formulas.

Examples:

1. given a classical propositional theory $\mathcal{T}$, the set of formulas modulo $\mathcal{T}$-interprovability is a Boolean algebra: $x \vee \neg x = \top$, ...
2. intuitionistic propositional logic: Heyting algebras: $x \to x = \top$, ...

The algebras of logic forget some syntactic notions.

E.g,: in the Boolean algebra obtained from a classical propositional theory, we cannot distinguish the "pure propositional variables" from the other formulas obtained as Boolean combinations of them.

At times, syntactic information may come in handy: in this talk, I will present one way to talk about quantifier-freeness and quantifier-alternation depth of formulas in the algebras of classical first-order logic.

Aim: extend *step-by-step methods* (already used in algebras of propositional logics) to algebras of logics with quantifiers.

Goal: develop step-by-step methods for nested quantifiers in the algebras of first-order classical logic.

Algebras of first-order classical logic = *first-order Boolean doctrines* (Lawvere, '60s).

For simplicity, we consider languages with only relational symbols (i.e. no function symbols), and mono-sorted.

Let $\mathcal{T}$ be a first-order theory in a purely relational language.

1. (Algebra of formulas in each context:) For each tuple $\underline{x} = \langle x_1, \dots, x_n \rangle$ of distinct variables (also called "context"), we have a Boolean algebra

$$\mathbf{LT}^{\mathcal{T}}(\underline{x})$$

(LT stands for "Lindenbaum-Tarski algebra") obtained by modding the set of formulas with free (possibly dummy) variables $x_1, \dots, x_n$ by $\mathcal{T}$-interprovability.

2. (Substitutions:) Given two contexts (= tuples of distinct variables) $\underline{x} = \langle x_1, \dots, x_n \rangle$ and $\underline{y} = \langle y_1, \dots, y_m \rangle$ and given a function

$$\sigma \colon \{x_1, \dots, x_n\} \to \{y_1, \dots, y_m\},$$

we have a function

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}_{\sigma} \colon \mathbf{LT}^{\mathcal{T}}(\underline{x}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\underline{y}) \\ \alpha(\underline{x}) &\longmapsto [\alpha(\underline{x}), x_i \mapsto \sigma(x_i)]. \end{aligned}$$

3. (Quantifiers:) For all tuples of distinct variables $\underline{x}$ and $\underline{y}$, we have functions

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}(\underline{x}, \underline{y}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\underline{x}) \\ \alpha(\underline{x}, \underline{y}) &\longmapsto \forall \underline{y}\, \alpha(\underline{x}, \underline{y}) \end{aligned}$$

and

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}(\underline{x}, \underline{y}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\underline{x}) \\ \alpha(\underline{x}, \underline{y}) &\longmapsto \exists \underline{y}\, \alpha(\underline{x}, \underline{y}). \end{aligned}$$

The algebraic structure associated to the first-order theory $\mathcal{T}$ is captured by

1. (Algebra of formulas in each context:)

$$\mathbf{LT}^{\mathcal{T}}(\underline{x}).$$

2. (Substitutions:)

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}_{\sigma} \colon \mathbf{LT}^{\mathcal{T}}(\underline{x}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\underline{y}) \\ \alpha(\underline{x}) &\longmapsto [\alpha(\underline{x}), x_i \mapsto \sigma(x_i)]. \end{aligned}$$

3. (Quantifiers:)

$$\forall \underline{y}, \exists \underline{y} \colon \mathbf{LT}^{\mathcal{T}}(\underline{x}, \underline{y}) \to \mathbf{LT}^{\mathcal{T}}(\underline{x}).$$

What are the properties satisfied by the structures arising in this way?

Answer: first-order Boolean doctrines.

A *first-order Boolean doctrine* consists of

1. for each tuple $\underline{x}$ of distinct variables, a Boolean algebra $\mathbf{P}(\underline{x})$
   (interpretation: algebra of formulas with free variables in $\underline{x}$)
2. for all tuples $\underline{x}$ and $\underline{y}$ of distinct variables and every function $\sigma\colon \{x_1, \dots, x_n\} \to \{y_1, \dots, y_m\}$, a Boolean homomorphism

$$\mathbf{P}_\sigma \colon \mathbf{P}(\underline{x}) \to \mathbf{P}(\underline{y}),$$

   satisfying (as a family) functoriality: $\mathbf{P}_{\mathrm{id}} = \mathrm{id}$, $\mathbf{P}_{g \circ f} = \mathbf{P}_g \circ \mathbf{P}_f$.
   (interpretation: substitution)
3. for all tuples of distinct variables $\underline{x}$ and $\underline{y}$, a function

$$\forall \underline{y} \colon \mathbf{P}(\underline{x}, \underline{y}) \to \mathbf{P}(\underline{x})$$

   (from which the existential is definable)
   (interpretation: universal quantification)

satisfying the following properties:

1. Quantifiers are adjoint to dummization:
   for every $\alpha(\underline{x})$ and $\beta(\underline{x}, \underline{y})$,

$$\alpha(\underline{x}) \leq \forall \underline{y}\, \beta(\underline{x}, \underline{y}) \text{ in } \mathbf{P}(\underline{x}) \iff \alpha(\underline{x}, \underline{y}) \leq \beta(\underline{x}, \underline{y}) \text{ in } \mathbf{P}(\underline{x}, \underline{y}),$$

   where $\alpha(\underline{x}, \underline{y})$ is $\alpha(x)$ with $\underline{y}$ dummy.
2. Beck-Chevalley (substitution commutes with quantification over disjoint sets of variables):
   For every $\sigma \colon \{x_1, \dots, x_n\} \to \{x'_1, \dots, x'_m\}$, and $\alpha(\underline{x}, \underline{y})$,

$$\mathbf{P}_\sigma(\forall \underline{y}\, \alpha(\underline{x}, \underline{y})) = \forall \underline{y}\, \mathbf{P}_\sigma(\alpha(\underline{x}, \underline{y})),$$

   i.e.

$$[\forall \underline{y}\, \alpha(\underline{x}, \underline{y}), x_i \mapsto \sigma(x_i)] = \forall \underline{y}\, [\alpha(\underline{x}, \underline{y}), x_i \mapsto \sigma(x_i)].$$

First-order Boolean doctrines are precisely the algebras of classical first-order logic.

First-order Boolean doctrines	:	Classical first-order logic
	=	
Boolean algebras	:	Classical propositional logic

First-order Boolean doctrines can be thought of as many-sorted algebras (one sort for each context), or certain functors.

First-order Boolean doctrines forget syntactic information such as what formulas are quantifier-free.

E.g.:

1. theory of partial orders.
2. theory of partial orders with an additional unary relation symbol "min" and the axiom

$$\forall x\,(\mathrm{min}(x) \leftrightarrow (\forall y\, x \leq y)).$$

The notion of quantifier-free formulas is *not intrinsic* in a first-order Boolean doctrine.

Given a first-order theory $\mathcal{T}$, inside its Lindenbaum-Tarski algebra $\mathbf{LT}^{\mathcal{T}}$ lies the algebra of equivalence classes of quantifier-free formulas $\mathbf{LT}_0^{\mathcal{T}}$.

$\mathbf{LT}_0^{\mathcal{T}}$ is a Boolean doctrine.

A *Boolean doctrine* consists of

1. for each tuple $\underline{x}$ of distinct variables, a Boolean algebra $\mathbf{P}(\underline{x})$ (interpretation: set of formulas with free variables in $\underline{x}$)
2. for all tuples $\underline{x}$ and $\underline{y}$ of distinct variables and every function $f\colon \{x_1, \dots, x_n\} \to \{y_1, \dots, y_m\}$, a Boolean homomorphism

$$\mathbf{P}_\sigma\colon \mathbf{P}(\underline{x}) \to \mathbf{P}(\underline{y}),$$

satisfying (as a family) functoriality: $\mathbf{P}_{\mathrm{id}} = \mathrm{id}$, $\mathbf{P}_{g\circ f} = \mathbf{P}_g \circ \mathbf{P}_f$. (interpretation: substitution)

Definition

A *quantifier-free fragment* of a first-order Boolean doctrine $\mathbf{P}$ is a Boolean subdoctrine of $\mathbf{P}$ that generates $\mathbf{P}$.

Boolean subdoctrine: a subset of $\mathbf{P}$ closed under Boolean combinations and substitutions.

Generating: closing $\mathbf{P}_0$ under quantifiers, Boolean combinations, quantifiers, Boolean combinations... gives the whole $\mathbf{P}$.

Theorem

If $\mathbf{P}_0$ is a quantifier-free fragment of a first-order Boolean doctrine $\mathbf{P}$, then there is a theory $\mathcal{T}$ such that $\mathbf{P}$ is the Lindenbaum-Tarski algebra $\mathbf{LT}^{\mathcal{T}}$ of $\mathcal{T}$ and $\mathbf{P}_0$ consists precisely of (the equivalence classes of) quantifier-free formulas.

Once we have a quantifier-free fragment

$$\mathbf{P}_0 \subseteq \mathbf{P}$$

we can stratify formulas by quantifier alternation depth (= maximum depth of alternations of $\exists$ and $\forall$)

$$\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots \ \mathbf{P}$$

We next give an intrinsic axiomatization of the sequences

$$\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots$$

of "formulas stratified by quantifier alternation depth".

A *QA-stratified Boolean doctrine* is a sequence of Boolean doctrines (recall that these have Boolean operations and substitutions, but not quantifiers)

$$\mathbf{P}_0 \leq \mathbf{P}_1 \leq \mathbf{P}_2 \leq \dots$$

equipped, for all $\underline{x}$, $\underline{y}$, and every $n \in \mathbb{N}$, with a function

$$\forall \underline{y} \colon \mathbf{P}_n(\underline{x}, \underline{y}) \longrightarrow \mathbf{P}_{n+1}(\underline{x})$$

(interpretation: universal quantification) satisfying:

1. Quantifiers are adjoint to dummization:
   for every $n \in \mathbb{N}$, $\alpha(\underline{x}) \in \mathbf{P}_{n+1}(\underline{x})$ and $\beta(\underline{x}, \underline{y}) \in \mathbf{P}_{n}(\underline{x}, \underline{y})$:

$$\alpha(\underline{x}) \leq \forall \underline{y}\, \beta(\underline{x}, \underline{y}) \text{ in } \mathbf{P}_{n+1}(\underline{x}) \iff \alpha(\underline{x}, \underline{y}) \leq \beta(\underline{x}, \underline{y}) \text{ in } \mathbf{P}_{n+1}(\underline{x}, \underline{y}).$$

2. Beck-Chevalley (quantifiers commute with substitutions):
   For every $n \in \mathbb{N}$, $\sigma \colon \{x_1, \dots, x_n\} \to \{x'_1, \dots, x'_m\}$, and $\alpha(\underline{x}, \underline{y})$,

$$[\forall \underline{y}\, \alpha(\underline{x}, \underline{y}), x_i \mapsto \sigma(x_i)]_{n+1} = \forall \underline{y}\, [\alpha(\underline{x}, \underline{y}), x_i \mapsto \sigma(x_i)]_n$$

3. Generation:
   For all n and $\underline{x}$, the Boolean algebra $\mathbf{P}_{n+1}(\underline{x})$ is generated by

$$\{\forall \underline{y}\, \alpha(\underline{x}, \underline{y}) \mid \underline{y} \text{ tuple of distinct variables}, \alpha(\underline{x}, \underline{y}) \in \mathbf{P}_n(\underline{x}, \underline{y})\}.$$

Theorem

QA-stratified Boolean doctrines capture precisely (up to isomorphism) the sequences

$$\mathbf{LT}_0^{\mathcal{T}} \subseteq \mathbf{LT}_1^{\mathcal{T}} \subseteq \mathbf{LT}_2^{\mathcal{T}} \subseteq \dots$$

obtained from a first-order theory and stratifying the formulas by quantifier alternation depth.

Next question: for any $n \in \mathbb{N}$, what is the algebraic structure of the finite sequences of the form

$$\mathbf{LT}_0^{\mathcal{T}} \subseteq \mathbf{LT}_1^{\mathcal{T}} \subseteq \mathbf{LT}_2^{\mathcal{T}} \cdots \subseteq \mathbf{LT}_n^{\mathcal{T}}$$

arising from a first-order theory $\mathcal{T}$?

Conjecture: take the axioms of QA-stratified Boolean doctrine and consider only those not involving any element of $\mathbf{LT}_k^{\mathcal{T}}$ for $k > n$.

We did the easiest case: $n = 0$.

This gives Boolean doctrines.

Theorem

Boolean doctrines are precisely the structures $\mathbf{P}_0$ *that appear in some QA-stratified Boolean doctrine* $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2 \ldots)$*.*

Equivalently, they are precisely the structures that appear as quantifier-free fragments of some first-order Boolean doctrine.

The difficult direction says that, for every Boolean doctrine $\mathbf{P}_0$, there is a first-order theory $\mathcal{T}$ such that, for all $\underline{x}$, $\mathbf{P}_0(\underline{x})$ is the quotient of the set of quantifier-free formulas with free variables $\underline{x}$ modulo $\mathcal{T}$-interprovability.

Idea: $\mathbf{P}_0$ encodes a universal theory. That's the theory. In fact, this is the "free way" in which to obtain a QA-stratified Boolean doctrine $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \ldots)$ from $\mathbf{P}_0$.

Theorem

The forgetful functor

$$\textit{first-order Boolean doctrines} \longrightarrow \textit{Boolean doctrines}$$

(which forgets quantifiers) has a left adjoint (which freely adds quantifiers).

Let $\mathbf{P}_0$ be a Boolean doctrine $\mathbf{P}_0$, let $\mathbf{P}^{\mathrm{free}}$ be the first-order Boolean doctrine obtained by freely adding quantifiers.

The comparison map

$$\mathbf{P}_0 \to \mathbf{P}^{\mathrm{free}}$$

is injective and generating: $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}^{\mathrm{free}}$.

Then one can stratify $\mathbf{P}^{\mathrm{free}}$ based on quantifier-alternation depth.

$$\mathbf{P}_0 \hookrightarrow \mathbf{P}_1^{\mathrm{free}} \hookrightarrow \mathbf{P}_2^{\mathrm{free}} \hookrightarrow \dots \; \mathbf{P}^{\mathrm{free}}$$

where $\mathbf{P}_n^{\mathrm{free}}$ collects the formulas of QAD $\leq n$.

We show how to explicitly construct $\mathbf{P}_1^{\text{free}}$ from $\mathbf{P}_0$. This is essentially a doctrinal version of Herbrand's theorem.

Theorem (Herbrand, 1930)

If $\mathcal{T}$ is a universal theory, and $\alpha(x)$ is quantifier-free, then

$$\vdash_{\mathcal{T}} \exists x \, \alpha(x)$$

holds if and only if there is a finite sequence of term-definable constants $c_1, \dots, c_k$ such that

$$\vdash_{\mathcal{T}} \alpha(c_1) \vee \cdots \vee \alpha(c_k).$$

This admits a slightly more general formulation that characterizes when a formula of quantifier alternation depth ≤ 1 entails another formula of quantifier alternation depth ≤ 1 modulo a universal theory.

We prove a doctrinal version of Herbrand's theorem.

Technical details: this is true in general, also when there are function symbols and multiple sorts. We only require the category of contexts to be small ($\approx$ only a *set* of sorts), in order to guarantee that $\mathbf{P}^{\mathrm{free}}$ and $\mathbf{P}_1^{\mathrm{free}}$ exist. The same construction works also with equality.

Theorem (Doctrinal version of Herbrand's theorem for formulas with QAD ≤ 1)

Let $\mathbf{P} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine, with* C *small, let* $\mathbf{P}^{\mathrm{free}}$ *be its quantifier completion, let* $\mathbf{P}_1^{\mathrm{free}}$ *be the Boolean subdoctrine of* $\mathbf{P}^{\mathrm{free}}$ *of "formulas with QAD* ≤ 1*". Then ... [= way to transform inequalities between elements in* $\mathbf{P}_1^{\mathrm{free}}$ *into equivalent existence of terms such that certain inequalities hold in* $\mathbf{P}_0$*].*

This describes the algebra $\mathbf{P}_1^{\mathrm{free}}$ of formulas with quantifier alternation depth ≤ 1 freely constructed over $\mathbf{P}_0$.

Contributions:

1. We axiomatize what substructures of a given first-order Boolean doctrine can be "quantifier-free fragments".
2. We axiomatize the algebraic structure of the sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \mathbf{LT}_2^{\mathcal{T}}, \dots)$ obtained from a first-order theory $\mathcal{T}$ by stratifying by QAD the $\mathcal{T}$-equivalence classes of formulas.
3. Boolean doctrines $=$ structures occurring as layer 0 in the sequences.
4. We obtain a doctrinal version of Herbrand's theorem for formulas with QAD ≤ 1.
   This describes how to freely construct the layer 1 (formulas with QAD ≤ 1) given the layer 0 (quantifier-free formulas).

M. Abbadini, F. Guffanti. Quantifier-free formulas and quantifier alternation depth in doctrines. On arXiv.

Future work:

1. Axiomatize the finite sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \dots, \mathbf{LT}_n^{\mathcal{T}})$ obtained from a first-order theory $\mathcal{T}$ by stratifying by QAD the $\mathcal{T}$-equivalence classes of formulas up to QAD n.

   We have done: $n = 0$.

   Enough: $n = 1$.

2. Show how to freely add a layer of QAD to one such sequence $(\mathbf{P}_0, \dots, \mathbf{P}_n)$ (without destroying the existing quantifiers).

   We have done: $n = 0$.

   Enough: $n = 1$.

   I.e.: let $\mathcal{T}$ be a theory whose axioms are universal closures of formulas of QAD ≤ 1. Give a criterion for when a given formula of QAD ≤ 2 is provable from $\mathcal{T}$, in terms of $\mathcal{T}$-provability of formulas of QAD ≤ 1.

Contributions:

1. Axiomatize "quantifier-free fragments".
2. Axiomatize the sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \mathbf{LT}_2^{\mathcal{T}}, \dots)$ obtained from a first-order theory $\mathcal{T}$ by stratifying by QAD the $\mathcal{T}$-equivalence classes of formulas.
3. Boolean doctrines = structures occurring as layer 0 in the sequences.
4. Doctrinal version of Herbrand's theorem for formulas with QAD ≤ 1. This describes how to freely add the first layer of QAD to a Boolean doctrine.

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Future work:

1. Axiomatize the finite sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \dots, \mathbf{LT}_n^{\mathcal{T}})$.
2. Describe how to freely add a layer of QAD to $(\mathbf{P}_0, \dots, \mathbf{P}_n)$.

Thank you!