Quantifier-free fragments and quantifier alternation depth in doctrines

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Based on joint works with Francesca Guffanti:

- Quantifier-free formulas and quantifier alternation depth in doctrines. On arXiv.
- Freely adding one layer of quantifiers to a Boolean doctrine. On arXiv.

The algebras of logic forget some syntactic information.

E.g.: for a classical propositional theory $\mathcal{T}$, in the Boolean algebra of propositional formulas modulo $\mathcal{T}$-interprovability we cannot distinguish the "pure propositional variables" from the other formulas obtained from them via Boolean combinations.

The algebras of classical first-order theory: we cannot distinguish the quantifier-free formulas from the other ones, we cannot talk about quantifier alternation depth.

I will present one way to talk about *quantifier-free* formulas and *quantifier alternation depth* in the algebras of classical first-order logic.

Inspiration: step-by-step methods in propositional modal logic and intuitionistic propositional logic.

First-order classical logic

Goal: develop step-by-step methods for nested **quantifiers** in the algebras of first-order classical logic.

Algebras of first-order classical logic = *first-order Boolean doctrines* (Lawvere, '60s).

For simplicity, in this talk, we consider only mono-sorted languages with only relational symbols (i.e. no function symbols) and no equality predicate.

Let $\mathcal{T}$ be a first-order theory.

1. (Algebra of formulas in each context:) For each tuple $\overline{x} = \langle x_1, \ldots, x_n \rangle$ of distinct variables (also called "context"), we have a Boolean algebra

$$\mathbf{LT}^{\mathcal{T}}(\overline{x})$$

(LT stands for "Lindenbaum-Tarski algebra") obtained by modding by $\mathcal{T}$-interprovability the set of formulas with free (possibly dummy) variables $x_1, \ldots, x_n$.

2. (Substitutions:) Given two contexts (= tuples of distinct variables) $\overline{x} = \langle x_1, \ldots, x_n \rangle$ and $\overline{y} = \langle y_1, \ldots, y_m \rangle$ and given a function

$$\sigma \colon \{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_m\},$$

we have a function

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}_{\sigma} \colon \mathbf{LT}^{\mathcal{T}}(\overline{x}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\overline{y}) \\ \alpha(\overline{x}) &\longmapsto [\alpha(\overline{x}), x_i \mapsto \sigma(x_i)]. \end{aligned}$$

3. (Quantifiers:) For all tuples of distinct variables $\overline{x}$ and $\overline{y}$, we have functions

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}(\overline{x},\overline{y}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\overline{x}) \\ \alpha(\overline{x},\overline{y}) &\longmapsto \forall \overline{y}\, \alpha(\overline{x},\overline{y}) \end{aligned}$$

and

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}(\overline{x},\overline{y}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\overline{x}) \\ \alpha(\overline{x},\overline{y}) &\longmapsto \exists \overline{y}\, \alpha(\overline{x},\overline{y}). \end{aligned}$$

The algebraic structure associated to the first-order theory $\mathcal{T}$ is captured by

1. (Algebra of formulas in each context:)

$$\mathbf{LT}^{\mathcal{T}}(\overline{x}).$$

2. (Substitutions:)

$$\begin{aligned} \mathbf{LT}^{\mathcal{T}}_{\sigma} \colon \mathbf{LT}^{\mathcal{T}}(\overline{x}) &\longrightarrow \mathbf{LT}^{\mathcal{T}}(\overline{y}) \\ \alpha(\overline{x}) &\longmapsto [\alpha(\overline{x}), x_i \mapsto \sigma(x_i)]. \end{aligned}$$

3. (Quantifiers:)

$$\forall \overline{y}, \exists \overline{y} \colon \mathbf{LT}^{\mathcal{T}}(\overline{x}, \overline{y}) \to \mathbf{LT}^{\mathcal{T}}(\overline{x}).$$

Structures arising in this way: first-order Boolean doctrines (Lawvere, '60s).

A *first-order Boolean doctrine* consists of

1. for each tuple $\overline{x}$ of distinct variables, a Boolean algebra $\mathbf{P}(\overline{x})$
   (interpretation: algebra of formulas with free variables in $\overline{x}$)
2. for all tuples $\overline{x}$ and $\overline{y}$ of distinct variables and every function $\sigma\colon \{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_m\}$, a Boolean homomorphism

   $$\mathbf{P}_\sigma\colon \mathbf{P}(\overline{x}) \to \mathbf{P}(\overline{y}),$$

   satisfying (as a family) functoriality: $\mathbf{P}_{\mathrm{id}} = \mathrm{id}$, $\mathbf{P}_{g \circ f} = \mathbf{P}_g \circ \mathbf{P}_f$.
   (interpretation: substitution)
3. for all tuples of distinct variables $\overline{x}$ and $\overline{y}$, a function

   $$\forall \overline{y}\colon \mathbf{P}(\overline{x}, \overline{y}) \to \mathbf{P}(\overline{x})$$

   (from which the existential is definable)
   (interpretation: universal quantification)

satisfying the following properties:

1. Quantifiers are adjoint to dummization:
   for every $\alpha(\overline{x})$ and $\beta(\overline{x},\overline{y})$,

$$\alpha(\overline{x}) \leq \forall \overline{y}\, \beta(\overline{x},\overline{y}) \text{ in } \mathbf{P}(\overline{x}) \iff \alpha(\overline{x},\overline{y}) \leq \beta(\overline{x},\overline{y}) \text{ in } \mathbf{P}(\overline{x},\overline{y}),$$

   where $\alpha(\overline{x},\overline{y})$ is $\alpha(x)$ with $\overline{y}$ dummy.
2. Beck-Chevalley (substitution commutes with quantification over disjoint sets of variables):
   For every $\sigma \colon \{x_1,\dots,x_n\} \to \{x'_1,\dots,x'_m\}$, and $\alpha(\overline{x},\overline{y})$,

$$\mathbf{P}_\sigma(\forall \overline{y}\, \alpha(\overline{x},\overline{y})) = \forall \overline{y}\, \mathbf{P}_\sigma(\alpha(\overline{x},\overline{y})),$$

   i.e.

$$[\forall \overline{y}\, \alpha(\overline{x},\overline{y}), x_i \mapsto \sigma(x_i)] = \forall \overline{y}\, [\alpha(\overline{x},\overline{y}), x_i \mapsto \sigma(x_i)].$$

First-order Boolean doctrines are precisely the algebras of classical first-order logic.

$$\frac{\text{First-order Boolean doctrines} \quad : \quad \text{Classical first-order logic}}{\text{Boolean algebras} \quad : \quad \text{Classical propositional logic}}$$

First-order Boolean doctrines can be thought of as many-sorted algebras (one sort for each context), or certain functors.

The actual definition of a first-order Boolean doctrine is a bit more general, capturing also the case of many-sorted languages and the presence of function symbols. It is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ where C is a category with finite products, satisfying certain properties...

First-order Boolean doctrines forget syntactic information such as which formulas are quantifier-free.

E.g.:

1. theory of partial orders.
2. theory of partial orders with an additional unary relation symbol "min" and the axiom

$$\forall x\,(\mathrm{min}(x) \leftrightarrow (\forall y\, x \leq y)).$$

The notion of quantifier-free formulas is *not intrinsic* in a first-order Boolean doctrine.

The main goals:

1. To introduce in the doctrinal approach enough structure to track quantifier-free formulas, quantifier alternation depth.
2. To give an explicit construction of free algebras by freely adding quantifiers.

Quantifier-free formulas

Given a first-order theory $\mathcal{T}$, inside its Lindenbaum-Tarski algebra $\mathbf{LT}^{\mathcal{T}}$ lies the algebra of equivalence classes of quantifier-free formulas $\mathbf{LT}_0^{\mathcal{T}}$.

$\mathbf{LT}_0^{\mathcal{T}}$ is a Boolean doctrine.

A *Boolean doctrine* consists of

1. for each tuple $\overline{x}$ of distinct variables, a Boolean algebra $\mathbf{P}(\overline{x})$ (interpretation: set of formulas with free variables in $\overline{x}$)
2. for all tuples $\overline{x}$ and $\overline{y}$ of distinct variables and every function $f\colon \{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_m\}$, a Boolean homomorphism

$$\mathbf{P}_\sigma\colon \mathbf{P}(\overline{x}) \to \mathbf{P}(\overline{y}),$$

satisfying (as a family) functoriality: $\mathbf{P}_{\mathrm{id}} = \mathrm{id}$, $\mathbf{P}_{g\circ f} = \mathbf{P}_g \circ \mathbf{P}_f$. (interpretation: substitution)

Definition

A *quantifier-free fragment* of a first-order Boolean doctrine $\mathbf{P}$ is a Boolean subdoctrine $\mathbf{P}_0$ of $\mathbf{P}$ that generates $\mathbf{P}$.

Boolean subdoctrine: a subset of $\mathbf{P}$ closed under Boolean combinations and substitutions.

Generating: closing $\mathbf{P}_0$ under quantifiers, Boolean combinations, quantifiers, Boolean combinations... gives the whole $\mathbf{P}$.

Theorem (MA, Guffanti)

If $\mathbf{P}_0$ is a quantifier-free fragment of a first-order Boolean doctrine $\mathbf{P}$, then there is a theory $\mathcal{T}$ such that $\mathbf{P}$ is the Lindenbaum-Tarski algebra $\mathbf{LT}^{\mathcal{T}}$ of $\mathcal{T}$ and $\mathbf{P}_0$ consists precisely of (the equivalence classes of) quantifier-free formulas.

This is analogous to:
"If X is a generating subset of a Boolean algebra B, then there is a propositional theory $\mathcal{T}$ whose Lindenbaum-Tarski Boolean algebra is B, and for which the elements of X are precisely the atomic propositional formulas."

Quantifier alternation depth

Given a quantifier-free fragment

$$\mathbf{P}_0 \subseteq \mathbf{P}$$

we can stratify formulas by quantifier alternation depth (= maximum depth of alternations of $\exists$ and $\forall$)

$$\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots \; \mathbf{P}$$

We next give an intrinsic axiomatization of the sequences

$$\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots$$

of "formulas stratified by quantifier alternation depth".

A *QA-stratified Boolean doctrine* is a sequence of Boolean doctrines (recall that these have Boolean operations and substitutions, but not quantifiers)

$$\mathbf{P}_0 \leq \mathbf{P}_1 \leq \mathbf{P}_2 \leq \dots$$

("$\leq$" means "embedding of Boolean doctrines") equipped, for all $\overline{x}$, $\overline{y}$, and every $n \in \mathbb{N}$, with a function

$$\forall \overline{y} \colon \mathbf{P}_{n}(\overline{x}, \overline{y}) \longrightarrow \mathbf{P}_{n+1}(\overline{x})$$

(interpretation: universal quantification)
satisfying:

1. Quantifiers are adjoint to dummization:
   for every $n \in \mathbb{N}$, $\alpha(\overline{x}) \in \mathbf{P}_{n+1}(\overline{x})$ and $\beta(\overline{x}, \overline{y}) \in \mathbf{P}_{n}(\overline{x}, \overline{y})$:

$$\alpha(\overline{x}) \leq \forall \overline{y}\, \beta(\overline{x}, \overline{y}) \text{ in } \mathbf{P}_{n+1}(\overline{x}) \iff \alpha(\overline{x}, \overline{y}) \leq \beta(\overline{x}, \overline{y}) \text{ in } \mathbf{P}_{n+1}(\overline{x}, \overline{y}).$$

2. Beck-Chevalley (quantifiers commute with substitutions):
   For every $n \in \mathbb{N}$, $\sigma \colon \{x_1, \dots, x_n\} \to \{x'_1, \dots, x'_m\}$, and $\alpha(\overline{x}, \overline{y})$,

$$[\forall \overline{y}\, \alpha(\overline{x}, \overline{y}), x_i \mapsto \sigma(x_i)]_{n+1} = \forall \overline{y}\, [\alpha(\overline{x}, \overline{y}), x_i \mapsto \sigma(x_i)]_{n}$$

3. Generation:
   For all n and $\overline{x}$, the Boolean algebra $\mathbf{P}_{n+1}(\overline{x})$ is generated by

$$\{\forall \overline{y}\, \alpha(\overline{x}, \overline{y}) \mid \overline{y} \text{ tuple of distinct variables, } \alpha(\overline{x}, \overline{y}) \in \mathbf{P}_{n}(\overline{x}, \overline{y})\}.$$

Theorem (MA, Guffanti)

QA-stratified Boolean doctrines capture precisely (up to isomorphism) the sequences

$$\mathbf{LT}_0^{\mathcal{T}} \subseteq \mathbf{LT}_1^{\mathcal{T}} \subseteq \mathbf{LT}_2^{\mathcal{T}} \subseteq \dots$$

obtained from a first-order theory and stratifying the formulas by quantifier alternation depth.

Quantifier alternation up to a fixed number

Question for future work: for any $n \in \mathbb{N}$, what is the algebraic structure of the finite sequences of the form

$$\mathbf{LT}_0^{\mathcal{T}} \subseteq \mathbf{LT}_1^{\mathcal{T}} \subseteq \mathbf{LT}_2^{\mathcal{T}} \cdots \subseteq \mathbf{LT}_n^{\mathcal{T}}$$

arising from a first-order theory $\mathcal{T}$?

Conjecture: take the axioms of QA-stratified Boolean doctrine and consider only those not involving any element of $\mathbf{LT}_k^{\mathcal{T}}$ for $k > n$.

We did the case $n = 0 \rightsquigarrow$ Boolean doctrines.

Theorem

Boolean doctrines are precisely the structures $\mathbf{P}_0$ *that appear in some QA-stratified Boolean doctrine* $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2 \dots)$.

Equivalently, they are precisely the structures that quantifier-free fragments of some first-order Boolean doctrine.

$(\subseteq)$ says: for every Boolean doctrine $\mathbf{P}_0$, there is a first-order theory $\mathcal{T}$ such that, for all $\overline{x}$, $\mathbf{P}_0(\overline{x})$ is the quotient modulo $\mathcal{T}$-interprovability of the set of quantifier-free formulas with free variables $\overline{x}$.

Idea: $\mathbf{P}_0$ encodes a universal theory. That's the theory. In fact, this is the "free way" in which to obtain a QA-stratified Boolean doctrine $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \dots)$ from $\mathbf{P}_0$.

Free construction

Theorem

The forgetful functor

$$\textit{first-order Boolean doctrines} \longrightarrow \textit{Boolean doctrines}$$

(which forgets quantifiers) has a left adjoint (which freely adds quantifiers).

Let $\mathbf{P}_0$ be a Boolean doctrine. We call *quantifier completion of* $\mathbf{P}_0$ the first-order Boolean doctrine $\mathbf{P}^{\mathrm{free}}$ obtained by freely adding quantifiers.

The comparison map

$$\mathbf{P}_0 \hookrightarrow \mathbf{P}^{\mathrm{free}}$$

is injective and generating: $\mathbf{P}_0$ is a quantifier-free fragment of $\mathbf{P}^{\mathrm{free}}$.

Then one can stratify $\mathbf{P}^{\mathrm{free}}$ based on quantifier-alternation depth.

$$\mathbf{P}_0 \hookrightarrow \mathbf{P}_1^{\mathrm{free}} \hookrightarrow \mathbf{P}_2^{\mathrm{free}} \hookrightarrow \dots \; \mathbf{P}^{\mathrm{free}}$$

where $\mathbf{P}_n^{\mathrm{free}}$ collects the formulas of QAD $\leq n$.

We show how to explicitly construct $\mathbf{P}_1^{\mathrm{free}}$ from $\mathbf{P}_0$. This is essentially a doctrinal version of Herbrand's theorem for formulas of QAD ≤ 1.

Theorem (Herbrand, 1930)

If $\mathcal{T}$ is a universal theory, and $\alpha(x)$ is quantifier-free, we have

$$\vdash_{\mathcal{T}} \exists x \, \alpha(x)$$

iff there are term-definable constants $c_1, \ldots, c_k$ s.t.

$$\vdash_{\mathcal{T}} \alpha(c_1) \vee \cdots \vee \alpha(c_k).$$

A more general formulation characterizes entailment between formulas of quantifier alternation depth ≤ 1 modulo a universal theory.

E.g.: Given a universal theory $\mathcal{T}$, we have

$$\forall y\, \alpha(y) \wedge \exists w\, \beta(w) \vdash_{\mathcal{T}} \forall z\, \gamma(z) \vee \exists v\, \delta(v)$$

(where α, β, γ and δ are quantifier-free) iff there are term-definable constants $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ such that

$$\bigwedge_{i=1}^{n} \alpha(c_i) \wedge \beta(w) \vdash_{\mathcal{T}} \gamma(z) \vee \bigvee_{k=1}^{m} \delta(d_k).$$

Theorem (Doctrinal version of Herbrand's theorem for formulas with QAD ≤ 1)

Let $\mathbf{P}_0 \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ *be a Boolean doctrine (= a functor), with* C *a small category with finite products. Then ...*

[= way to transform inequalities between "formulas of quantifier alternation depth ≤ 1" in the quantifier completion of $\mathbf{P}_0$ *into an equivalent existence of terms such that certain inequalities hold in* $\mathbf{P}_0$*].*

Theorem 4.9 (Doctrinal version of Herbrand's theorem). *Let $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ be a Boolean doctrine with C small, and let $(\mathrm{id}_{\mathsf{C}}, \mathbf{i})\colon \mathbf{P} \hookrightarrow \mathbf{P}^{\forall\exists}$ be a quantifier completion of $\mathbf{P}$. For all $S, Y_1, \dots, Y_{\bar{i}}, W_1, \dots, W_{\bar{h}}, Z_1, \dots, Z_{\bar{j}}, V_1, \dots, V_{\bar{k}} \in \mathsf{C}$, $(\alpha_i \in \mathbf{P}(S \times Y_i))_{i=1,\dots,\bar{i}}$, $(\gamma_h \in \mathbf{P}(S \times W_h))_{h=1,\dots,\bar{h}}$, $(\beta_j \in \mathbf{P}(S \times Z_j))_{j=1,\dots,\bar{j}}$ and $(\delta_k \in \mathbf{P}(S \times V_k))_{k=1,\dots,\bar{k}}$, the following conditions are equivalent.*

(1) In $\mathbf{P}^{\forall\exists}(S)$ we have

$$\left(\bigwedge_{i=1}^{\bar{i}} \forall_S^{Y_i} \mathbf{i}_{S\times Y_i}(\alpha_i)\right) \wedge \left(\bigwedge_{h=1}^{\bar{h}} \exists_S^{W_h} \mathbf{i}_{S\times W_h}(\gamma_h)\right) \leq \left(\bigvee_{j=1}^{\bar{j}} \forall_S^{Z_j} \mathbf{i}_{S\times Z_j}(\beta_j)\right) \vee \left(\bigvee_{k=1}^{\bar{k}} \exists_S^{V_k} \mathbf{i}_{S\times V_k}(\delta_k)\right).$$

(2) There are $n, n' \in \mathbb{N}$, $l_1, \dots, l_n \in \{1, \dots, \bar{i}\}$, $l'_1, \dots, l'_n \in \{1, \dots, \bar{k}\}$, $(g_i\colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to Y_{l_i})_{i=1,\dots,n}$ and $(g'_k\colon S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h \to V_{l'_k})_{k=1,\dots,n'}$ such that (in $\mathbf{P}(S \times \prod_{j=1}^{\bar{j}} Z_j \times \prod_{h=1}^{\bar{h}} W_h)$)

$$\bigwedge_{i=1}^{n} \mathbf{P}(\langle \mathrm{pr}_1, g_i\rangle)(\alpha_{l_i}) \wedge \bigwedge_{h=1}^{\bar{h}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{h+\bar{j}+1}\rangle)(\gamma_h) \leq \bigvee_{j=1}^{\bar{j}} \mathbf{P}(\langle \mathrm{pr}_1, \mathrm{pr}_{j+1}\rangle)(\beta_j) \vee \bigvee_{k=1}^{n'} \mathbf{P}(\langle \mathrm{pr}_1, g'_k\rangle)(\delta_{l'_k}).$$

This describes the (order on the) algebra $\mathbf{P}_1^{\mathrm{free}}$ of formulas with quantifier alternation depth ≤ 1 freely constructed over $\mathbf{P}_0$.

(It holds also with many sorts, with function symbols, with equality.)

Next steps, for future work: how to add further layers of quantifiers without destroying the quantifiers created at the previous steps.

This would give a step-by-step construction of the quantifier completion of a Boolean doctrine. It will amount to a doctrinal version of Herbrand's theorem for arbitrary first-order formulas.

Herbrand used the fact that every formula is equivalent to one in prenex normal form. I am not sure this holds in doctrines. In general, in doctrines:

$$(\exists x\, \alpha(x)) \vee \beta \neq \exists x\, (\alpha(x) \vee \beta).$$

E.g.: when $\alpha = \top$ and $\beta = \top$:

$$\top \neq \exists x\, \top(x).$$

Conclusions

The main goals:

1. To introduce in the doctrinal approach enough structure to track quantifier-free formulas, quantifier alternation depth.
2. To give an explicit construction of free algebras by freely adding quantifiers.

Contributions:

1. Axiomatize "quantifier-free fragments".
2. Axiomatize the sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \mathbf{LT}_2^{\mathcal{T}}, \dots)$ obtained from a theory $\mathcal{T}$ by stratifying by QAD the $\mathcal{T}$-equiv. classes of formulas.
3. Boolean doctrines = structures occurring as layer 0 in the sequences.

MA, F. Guffanti. Quantifier-free formulas and quantifier alternation depth in doctrines. (arXiv.)

4. How to freely add the first layer of QAD to a Boolean doctrine, i.e. doctrinal version of Herbrand's theorem for formulas with QAD ≤ 1.

MA, F. Guffanti. Freely adding one layer of quantifiers to a Boolean doctrine. (arXiv.)

Future work:

a. Axiomatize the finite sequences $(\mathbf{LT}_0^{\mathcal{T}}, \mathbf{LT}_1^{\mathcal{T}}, \dots, \mathbf{LT}_n^{\mathcal{T}})$.
b. Describe how to freely add a layer of QAD to $(\mathbf{P}_0, \dots, \mathbf{P}_n)$.
c. Do all of these dually (via Stone duality).

Thank you!