Coalgebraic flavour of metric compact Hausdorff spaces

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Compact metric spaces

A metric is a function $d\colon X \times X \to [0, \infty)$ s.t.

1. (Symmetry) $d(x, y) = d(y, x)$.
2. (Reflexivity) $d(x, x) = 0$.
3. (Separatedness) $d(x, y) = 0$ implies $x = y$.
4. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A *compact metric space* is a metric space that is compact in its induced topology. (It is necessarily Hausdorff.)

Example: $[0, 1]^n$.

The category of compact metric spaces (and non-expansive maps) is not great.

Can we tweak to the class of objects to get a good one?

The category of compact metric spaces and non-expansive maps:

Problem 1. **not cocomplete**: no coproduct of two singletons.

Remedy: allow distance ∞.

Definition

A *metric* (in this talk) is a map $d \colon X \times X \to [0, \infty]$ s.t.

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- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

The category of compact metric spaces and non-expansive maps:

Problem 2. **not complete**: no countable power of a two-element metric space.

Reason: the topology induced by the product metric (= sup metric) is **not** the product topology, and is not compact.

Remedy: have the topology **not necessarily induced** by the metric, but just compatible.

Instead of compact metric spaces...

Definition (Hofmann, Reis, 2018)

A *metric compact Hausdorff space* X is a compact Hausdorff space X together with a metric $d\colon X \times X \to [0, \infty]$ that is lower semicontinuous, i.e. that is continuous with respect to the upper topology of $[0, \infty]$.

(They did not require symmetry and separatedness.)

Upper topology of $[0, \infty]$: generated by $(u, \infty]$, $u \in [0, \infty]$.

Lower semicontinuity of d can also be expressed as:

$$d(x_0, y_0) \leq \liminf_{\substack{x \to x_0 \\ y \to y_0}} d(x, y).$$

Small perturbations may yield great increments in distances, but not great decrements.

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Example: any compact metric space.

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Example: any compact Hausdorff space where every pair of distinct points has distance 1.

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Example: Any product of compact metric spaces, with the product metric (= sup metric) and the product topology.

Theorem (Essentially Tholen 2009)

The category **MetCH** *of metric compact Hausdorff spaces and non-expansive continuous maps is complete and cocomplete.*

What other good categorical properties?

What about a well-behaved factorization system?

In most classes of algebras (groups, rings, monoids, Boolean algebras...),

1. morphisms have a (surjection, injection) factorization;
2. surjective morphisms = regular epis = strong epis;
3. injective morphisms = monos;
4. surjective morphisms are stable under pullbacks.

Closed under arbitrary products and subalgebras ⇒ regular.

Definition

A category $\mathcal{C}$ with finite limits is *regular* if

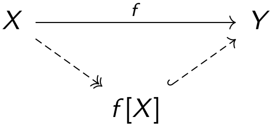
1. every morphism has a (strong epi, mono) factorization;
2. the pullback of a strong epi along any morphism is a strong epi.

In any regular category, strong epi = regular epi.

Closed under products, subalgebras, *homomorphic images* ⇒ Barr-exact.

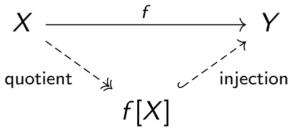
The category of topological spaces is **not** regular.

A continuous function has different (surjection, injection) factorizations.



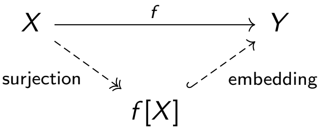
The two "extremal" cases are: equip $f[X]$ with the...

quotient topology induced by X:



(*strong epi*, mono) factorization.

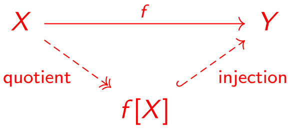
subspace topology induced by Y:



(epi, *strong mono*) factorization.

Strong epis (= quotients maps) are **not** stable under pullbacks (see a counterexample in Borceaux' HCA2). Therefore, **Top** is **not** regular.

quotient topology induced by X:



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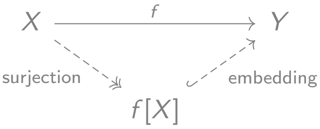
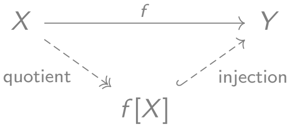


(epi, *strong mono*) factorization.

However, **Top** is **co**regular (e.g. [Barr, Pedicchio, 1995]): strong monos (= embeddings) are stable under pushouts.

quotient topology induced by X:



(*strong epi*, mono) factorization.

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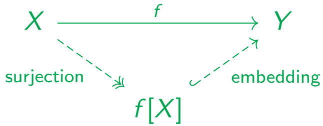


(epi, *strong mono*) factorization.

Definition

A category $\mathcal{C}$ with finite colimits is *coregular* if

1. every morphism has an (epi, strong mono) factorization;
2. the pushout of a strong mono along any morphism is a strong mono.

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \hookrightarrow & Y' \end{array}$$

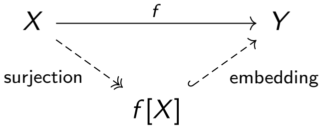
Coregularity ≈ given an embedding $X \hookrightarrow Y$ and a morphism $X \to X'$, we can replace X by X' inside Y.

Rough rule of thumb:

1. Categories of "algebras" are regular (and maybe even Barr-exact).
2. Categories of "spaces" are coregular (and maybe even Barr-coexact).

Metric compact Hausdorff spaces fall in the second class.

Given a morphism $f: X \to Y$ in **MetCH**, we equip $f[X]$ with the metric and topology induced by Y:



(epi, *strong mono*) factorization.

Theorem

In **MetCH**

1. *surjective morphisms = epis;*
2. *embeddings = strong monos = regular monos.*

Theorem

MetCH *is* *<u>co</u>regular.*

I.e.: embeddings are stable under pushouts.

Example: walking distance + add a train connection.

$$d(\text{Milan}, \text{Rome}) = 138 \qquad d'(\text{Milan}, \text{Rome}) = 3$$
$$d(\text{Milan}, \text{Naples}) = 184 \qquad d'(\text{Milan}, \text{Naples}) = 54$$
$$d(\text{Rome}, \text{Naples}) = 51 \qquad d'(\text{Rome}, \text{Naples}) = 51$$

Theorem

MetCH *is Barr-<u>co</u>exact.*

Roughly speaking, Barr-coexactness means:

For every surjective morphism $X + X \twoheadrightarrow Z$ satisfying certain properties ("reflexivity", "symmetry", "transitivity"), there is a closed subspace $A \subseteq X$ s.t. Z is obtained from $X + X$ by identifying, for each $y \in A$, the two copies of y in $X + X$.

We sketch the proof of the theorem.

Encoding of surjective morphisms: a surjective morphism

$$f: X \twoheadrightarrow Y,$$

can be encoded by the function

$$\begin{aligned} \gamma_f : X \times X &\longrightarrow [0, \infty] \\ (x_1, x_2) &\longmapsto d_Y(f(x_1), f(x_2)). \end{aligned}$$

Properties of γ_f?

1. It is a *premetric* (= symmetric, reflexive, triangle inequality, but distinct points may have distance 0). Moreover, it is *below* d_X, i.e.

   $$\gamma(x_1, x_2) \leq d_X(x_1, x_2).$$

   We say that it is a *sub-premetric* of X.
2. It is lower semicontinuous (= contin. wrt the upper top. of $[0, \infty]$).

Theorem

The isomorphism classes of surjective morphisms out of an object $X \in$ **MetCH** *are in bijective correspondence with the* **lower semicontinuous sub-premetrics on** X*.*

MetCH Barr-coexact: every internal equivalence corelation is a cokernel pair.

Given $X \in \mathbf{MetCH}$ and $A \hookrightarrow X$ an embedding.

The **cokernel pair** of $A \hookrightarrow X$ is the quotient $X + X \twoheadrightarrow (X + X)/{\sim}$, where $\sim$ identifies, for each $a \in A$, the two copies of a.

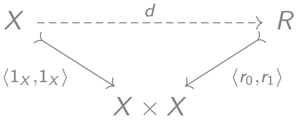
This quotient has an associated lower semicontinuous sub-premetric

$$(X + X) \times (X + X) \to [0, \infty].$$

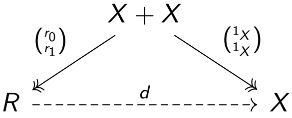
Internal binary corelation = equivalence class of a surjective $X + X \twoheadrightarrow Z$.
Equivalently, a lower semicontinuous sub-premetric γ on $X + X$.

- Reflexive: for all $i, j \in \{0, 1\}$ $d_X(x, y) \leq \gamma((x, i), (y, j))$.
- Symmetric: $\gamma((x, 0), (y, 1)) = \gamma((x, 1), (y, 0))$.
- Transitive: (under reflexivity and symmetry:) for all $x, y \in X$ with $\gamma((x, 0), (y, 1)) \neq \infty$ there is $z \in X$ such that $\gamma((x, 0), (y, 1)) = \gamma((x, 0), (z, 1)) + \gamma((z, 0), (y, 1))$.

Reflexive internal binary relation:



Reflexive internal binary **co**relation:



Theorem

For such a γ, for all $x, y \in X$ with $\gamma((x,0),(y,1)) \neq \infty$ there is $z \in X$ s.t. $\gamma((z,0),(z,1)) = 0$ and

$$\gamma((x,0),(y,1)) = d_X(x,z) + d_X(z,y).$$

Consequence: γ is the cokernel pair of the embedding $\{z \in X \mid \gamma((z,0),(z,1)) = 0\} \hookrightarrow X$.

Compactness is used crucially.

This conclues the sketch of the proof that the category **MetCH** of metric compact Hausdorff spaces is Barr-coexact.

To sum up

Allowing ∞ measures and topology not induced by the metric
$\rightsquigarrow$ great categorical properties:

The category **MetCH** of metric compact Hausdorff spaces and continuous non-expansive maps is <u>complete</u>, <u>cocomplete</u>, <u>Barr-coexact</u>.

- Open question: is **MetCH** dually equivalent to a variety of possibly infinitary algebras?
  Equivalently: does **MetCH** have a regular injective regular cogenerator?
  (If so, then function symbols of *infinite* arity are probably necessary.)
- Requiring the compact Hausdorff topology to be <u>induced</u> by the metric $\rightsquigarrow$ no infinite products.
  But is it still finitely cocomplete finitely complete Barr-coexact?
- Other quantales?

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