Barr-coexactness of metric compact Hausdorff spaces

Marco Abbadini

School of Computer Science, University of Birmingham, UK

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What's the opposite of

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It's

!

How is a category with a

good

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Various categories of "spaces" have a *good* algebraic opposite:

Category of spaces	Dual
Stone spaces	Boolean algebras [Stone, 1936]]
Priestley spaces (Stone + partial order)	Bdd. distr. lattices [Priestley, 1970]
Compact Hausdorff spaces	Monadic over **Set** [Duskin, 1969]
Compact ordered spaces (Comp. Hausd. + partial order) [Nachbin, '40s]	Monadic over **Set** [A., 2019]

From order to metric

Theorem (A., 2019)

The opposite of the category of

compact ordered spaces [Nachbin, '40s] $:=$
compact Hausdorff spaces $+$ *compatible partial order*

is monadic over **Set**.

"*Partial order*" is similar to "*metric*" [Lawvere].

Question

Is the opposite of the category of

compact Hausdorff spaces $+$ compatible *metric*

monadic over **Set**?

Is the **opposite** of the category of

compact Hausdorff spaces + compatible *metric*

good ?

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From a different angle

Compact metric space := metric space with compact (necessarily Hausdorff) induced topology.

Example: $[a, b]^n$, as well as any closed subspace.

The category of compact metric spaces and (necessarily continuous) non-expansive maps is poorly behaved.

- Can we tweak the category of *compact metric spaces* to get a good category?

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- Can we tweak the category of *compact metric spaces* to get a *good* category?

Compare it with

- Is the category of "*compact Hausd. spaces + compatible metric*" *good*?

Problems of compact metric spaces

The category of compact metric spaces and non-expansive maps...

Problem 1. ... is **not cocomplete**: no coproduct of two singletons.

Reason: we should put the maximum distance between the two points.

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Definition

A *metric* (in this talk) is a map $d\colon X \times X \to [0, \infty]$ satisfying:

- (symmetry) $d(x, y) = d(y, x)$;
- (reflexivity) $d(x, x) = 0$;
- (separatedness) $d(x, y) = 0$ implies $x = y$;
- (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

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Problem 2. ... is **not complete**: no countable power of a two-element metric space.

Reason: the topology induced by the product metric (= sup metric) is **not** the product topology (and is not compact).

Problems of compact metric spaces

The category of compact metric spaces and non-expansive maps...

Problem 2. ... is **not complete**: no countable power of a two-element metric space.

Reason: the topology induced by the product metric (= sup metric) is **not** the product topology (and is not compact).

Remedy: allow the topology to be just **compatible** with the metric, rather than **induced** by it.

What does “compatible” mean?

If

$$(f_n \colon X \to [a, b])_n$$

converges pointwise (i.e. in the product topology) to

$$f \colon X \to [a, b],$$

and $g \colon X \to [a, b]$ is another function,

$$d_\infty(f, g) \leq \liminf_{n \to \infty} d_\infty(f_n, g).$$

Definition (Hofmann, Reis, 2018)

A *metric compact Hausdorff space* is a compact Hausdorff space X equipped with a lower semicontinuous metric $X \times X \to [0, \infty]$.

Lower semicontinuous:

$$d(x_0, y_0) \leq \liminf_{\substack{x \to x_0 \\ y \to y_0}} d(x, y).$$

I.e.: small topological perturbations may yield great increments in distances, but not great decrements.

Equivalently, continuous wrt the upper topology of $[0, \infty]$.

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Example: any compact metric space (in the classical sense).

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Example: Any arbitrary product of compact metric spaces, with the product metric (= sup metric) and the product topology.

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Example: any compact Hausdorff space with $d(x, y) = 1$ for $x \neq y$.

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(Replacing "metric" by "order", we get compact ordered spaces.)

Category of metric compact Hausdorff spaces and continuous non-expansive maps =: **MetCH**.

Is $\mathbf{MetCH}^{\mathrm{op}}$ monadic over **Set**?

I.e.:

1. Is $\mathbf{MetCH}^{\mathrm{op}}$ complete and cocomplete?
2. Is $\mathbf{MetCH}^{\mathrm{op}}$ Barr-exact?
3. Does $\mathbf{MetCH}^{\mathrm{op}}$ have a regular projective regular generator?

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   ✓(Essentially: Tholen, 2009)

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1. Categories of "algebras": the *good* factorization tends to be "(regular epi, mono)" $\rightsquigarrow$ regularity.
2. Categories of "spaces": the *good* factorization tends to be "(epi, regular mono)" $\rightsquigarrow$ <u>co</u>regularity.

Example: **Top** is...

- ... *not* regular (proof in Borceaux' HCA2).
- ... <u>co</u>regular [Barr, Pedicchio, 1995].

1. Categories of “algebras”: the *good* factorization tends to be “(regular epi, mono)” $\rightsquigarrow$ regularity.
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Example: **Top** is...

- ... *not* regular (proof in Borceaux' HCA2).
- ... <u>co</u>regular [Barr, Pedicchio, 1995].

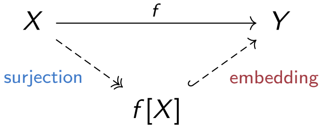
3. Categories of “(compact Hausdorff)-ish spaces” tend to be Barr-<u>co</u>exact.

Example:

- **Top** is *not* Barr-<u>co</u>exact.
- **CompHaus** is Barr-<u>co</u>exact [Duskin, 1969] (and [Barr & Wells, 1984] for more details).

In metric compact Hausdorff spaces

Given a morphism $f: X \to Y$ in **MetCH**, we equip $f[X]$ with the metric and topology induced by Y:



(epi, regular mono) factorization.

Theorem (A., Hofmann, 2025)

In the category **MetCH** *of metric compact Hausdorff spaces,*

1. *epis = surjective morphisms;*
2. *regular monos = strong monos = embeddings.*

Theorem (A., Hofmann, 2025)

MetCH *is* *<u>co</u>regular.*

$$\begin{array}{ccc} K & \hookrightarrow & X \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ K' & \hookrightarrow & X' \end{array}$$

Given an embedding $K \hookrightarrow X$ and a morphism $f\colon K \to K'$, inside X we can replace K by an exact copy of K' (and appropriate adjustments outside of K' induced by f).

Theorem (A., Hofmann, 2025)

MetCH *is Barr-co**exact.*

Given a metric compact Hausdorff space X. For $K \subseteq X$ closed $\rightsquigarrow$ quotient of $X + X$ by gluing the two copies of K.

Barr-coexactness: every surjective morphism $X + X \twoheadrightarrow Z$ satisfying "co-reflexivity", ~~"co-symmetry"~~, "co-transitivity" (first-order conditions) arises in this way.

MetCH $:=$ category of metric compact Hausdorff spaces and continuous non-expansive maps. [Hofmann, Reis, 2019]

- Is $\mathbf{MetCH}^{\mathrm{op}}$ monadic over **Set**? (I.e., equivalent to a variety of possibly infinitary algebras)? I.e.:
  1. Is **MetCH** complete and cocomplete? ✓ (Tholen, 2009)
  2. Is **MetCH** Barr-coexact? ✓ (A., Hofmann, 2025)
  3. Does **MetCH** have a regular inj. reg. cogenerator?

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  1. **MetCH** is not bad.

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- Can we tweak the category of classical "compact metric spaces" to get a *good* one?

1. **MetCH** is not bad.

1+2. **MetCH** is quite *good*!