Freely adding a layer of Heyting implication to a distributive lattice

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Work in progress
with Rodrigo Almeida and Igor Arrieta

Describe free Heyting algebras.

Difficult.

Free algebra on 1 generator: infinite.

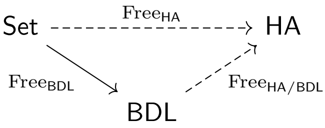
Free algebra on 2 generators: a lattice-theoretic description is notoriously difficult.

1. Ghilardi provided a *step-by-step* description of the free Heyting algebra over a finite set $\{x_1, \ldots, x_n\}$.
   - First describe the algebra of formulas with implication depth 0: the free bounded distributive lattice generated by n elements.
   - Suppose you have a description for the algebra of formulas with implication depth at most k. Then I give you a description of the algebra of formulas with implication depth at most $k+1$.

   (Every free Heyting algebra over a finite set is a biHeyting algebra.)

   It's a dual description. (Birkhoff's dual. for finite bdd. distr. lattices.)

2. Rodrigo Almeida generalized it to free Heyting algebras over a set of arbitrary cardinality. And more generally, for the free Heyting algebra over a given bounded distributive lattice.



   It's a dual description. (Priestley duality for bdd. distr. lattices.)

Let's do it purely algebraically.

My motivation:

1. Describe the free algebras of first-order classical logic. (Algebraic version of Herbrand's theorem.) Describe how to freely add one layer of quantifier-alternation depth at a time.
2. Pitts' problem: is every Heyting algebra the algebra of truth values of some topos? Pataraia claimed to have a positive answer, using a construction that adds quantifiers iteratively.

Maybe the case of implications is analogous.

Let's go!

Let L be a bounded distributive lattice. We want to describe

$$\mathrm{Free}_{\mathsf{HA}/\mathsf{BDL}}(L).$$

This will be the set of all Heyting combinations of elements from L, modulo the axioms of Heyting algebras and the axioms saying that the bounded lattice operations in L have to be preserved.

Step-by-step: We start by describing the set

$$L \hookrightarrow \quad L_{\rightarrow} \quad \hookrightarrow \mathrm{Free}_{\mathsf{HA}/\mathsf{BDL}}(L).$$

of elements of implication-depth at most 1: i.e., bounded lattice combinations of elements of the form

$$a \rightarrow b$$

for $a, b \in L$. (This includes L itself, since $a = \top \rightarrow a$.)

From a logical perspective

Suppose you have a set X of propositional variables, and a set $\mathcal{T}$ (a *theory*) of entailments

$$\varphi(x_1, \dots, x_n) \vdash \psi(x_1, \dots, x_m),$$

between bounded lattice combinations of elements from X.

We want to describe which entailment of the form

$$\bigwedge_{i=1}^{n}(a_i \to b_i) \vdash \bigvee_{j=1}^{m}(c_i \to d_i)$$

(with $a_i, b_i, c_j, d_j \in X$ and where $\to$ is an intuitionistic implication) follow.

In terms of what entailments

$$\varphi(x_1, \dots, x_n) \vdash \psi(x_1, \dots, x_m),$$

(between bdd. latt. combinations of elements from X) follow from $\mathcal{T}$.

Let L be a bounded distributive lattice. Let $L \hookrightarrow L_{\to}$ be the inclusion of L into the first layer of implication depth.

Example

In $L_{\to}$ we have

$$\top \to b \leq c \to d$$

if and only if in L...

$$b \wedge c \leq d.$$

Example

In $L_{\to}$ we have

$$a \to b \leq \bot$$

if and only if in L...

$$\begin{cases} a = \top, \\ b = \bot. \end{cases}$$

Example

In $L_{\to}$ we have

$$a \to b \leq d$$

if and only if in L...

$$\begin{cases} d \vee a = \top, \\ b \leq d. \end{cases}$$

Example

In $L_\to$ we have

$$a \to b \leq c \to d$$

if and only if in L

$$\begin{cases} c \leq a \vee d, \\ c \wedge b \leq d. \end{cases}$$

Example

In $L_{\rightarrow}$ we have

$$a \rightarrow \bot \leq c \rightarrow \bot$$

if and only if in L...

$$c \leq a.$$

Example

In $L_\to$ we have

$$a \to \bot \leq c \to d$$

if and only if in L

$$c \leq d \vee a.$$

In $L_{\to}$ we have

$$(a_1 \to b_1) \wedge b_2 \leq \bot$$

if and only if in L...

$$\begin{cases} b_2 \leq a_1, \\ b_1 \wedge b_2 = \bot. \end{cases}$$

In $L_{\to}$ we have

$$(a_1 \to b_1) \wedge (a_2 \to b_2) \leq \bot$$

if and only if in L...

$$\begin{cases} a_1 \vee a_2 = \top \\ b_1 \leq a_2 \\ b_2 \leq a_1 \\ b_1 \wedge b_2 = \bot \end{cases}$$

Theorem

In $L_\rightarrow$ we have

$$\bigwedge_{i=1}^{n}(a_i \rightarrow b_i) \leq c \rightarrow d$$

(with $a_i, b_i, c, d \in L$) iff, for all $S \subseteq \{1, \ldots, n\}$,

$$c \wedge \bigwedge_{i \in S} b_i \leq d \vee \bigvee_{i \in \{1,\ldots,n\} \setminus S} a_i.$$

I.e.: in the Boolean envelope of L

$$\bigwedge_{i=1}^{n}(a_i \rightsquigarrow b_i) \leq c \rightsquigarrow d.$$

Here, $\rightsquigarrow$ denotes the classical implication in the Boolean envelope of L.

Example

In $L_{\rightarrow}$ we have

$$\top \leq (c_1 \rightarrow \bot) \vee (c_2 \rightarrow \bot)$$

if and only if in $L_{\dots}$ there are e_1, e_2 such that

$$\begin{cases} \top = e_1 \vee e_2 \\ e_1 \wedge c_1 = \bot \\ e_2 \wedge c_2 = \bot \end{cases}$$

In $L_{\to}$ we have

$$\top \to b \leq (c_1 \to d_1) \vee (c_2 \to d_2)$$

if and only if in $L_{\dots}$ there are e_1, e_2 such that

$$\begin{cases} b \leq e_1 \vee e_2, \\ e_1 \wedge c_1 \wedge b \leq d_1, \\ e_2 \wedge c_2 \wedge b \leq d_2. \end{cases}$$

Theorem

In $L_\rightarrow$ we have

$$\bigwedge_{i=1}^{n} a_i \rightarrow b_i \leq \bigvee_{j=1}^{m} c_j \rightarrow d_j$$

(with $a_i, b_i, c_j, d_j \in L$) iff $\exists e_1, \dots, e_m \in L$ s.t., for all $S \subseteq \{1, \dots, n\}$,

$$\bigwedge_{i \in S} b_i \leq e_1 \vee \cdots \vee e_m \vee \bigvee_{i \in \{1,\dots,n\} \setminus S} a_i$$

$$\text{for all } j \qquad e_j \wedge c_j \wedge \bigwedge_{i \in S} b_i \leq d_j \vee \bigvee_{i \in \{1,\dots,n\} \setminus S} a_i$$

iff $\exists e_1, \dots, e_m \in L$ such that (in the Boolean envelope of L)

$$\bigwedge_{i=1}^{n} (a_i \rightsquigarrow b_i) \leq \bigvee_{j=1}^{m} e_j,$$

$$\text{for all } j \qquad e_j \leq \left(\bigwedge_{i=1}^{n} (a_i \rightsquigarrow b_i) \right) \rightsquigarrow (c_j \rightsquigarrow d_j).$$

If we have a lattice L, and we want to describe the free Heyting algebra H over L by describing layer by layer, what should we do?

We add freely the first layer of implications (theorem in the previous slide).

Then we add further layers preserving the previous implications (we are working now on this).

We describe how to add a layer of Heyting implication to a bounded distributive lattice, and then (hopefully) show how to continue iteratively.

Thank you!