On De Groot and Lawson self-dualities

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Work in progress with Achim Jung

Openness and compactness display strong signs of a certain symmetry. (As emphasised in [Jung & Sünderhauf, 1995].)

Definition (Compact)

A subset K of a space is *compact* if

$$\text{for every directed set } \mathcal{I} \text{ of open sets with } K \subseteq \bigcup \mathcal{I}$$
$$\text{there is } U \in \mathcal{I} \text{ such that } K \subseteq U.$$

Directed $:=$ nonempty $+$ any two elements have a common upper bound.

Fact

Let U be an open subset of a Hausdorff space. Then,

$$\textit{for every codirected set } \mathcal{F} \textit{ of compact sets with } \bigcap \mathcal{F} \subseteq U$$
$$\textit{there is } K \in \mathcal{F} \textit{ such that } K \subseteq U.$$

E.g.: if $\bigcap_n [a_n, b_n] \subseteq (-1, 1)$ with $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$, then there is n s.t. $[a_n, b_n] \subseteq (-1, 1)$.

Theorem

Let U be an open subset of a sober space. Then,

$$\text{for every codirected set } \mathcal{F} \text{ of compact saturated sets with } \bigcap \mathcal{F} \subseteq U \\ \text{there is } K \in \mathcal{F} \text{ such that } K \subseteq U.$$

Saturated := intersection of open sets = upset in the specialization order.

In T_1 spaces (e.g., in any Hausdorff space), every subset is saturated.

$$\text{open} \overset{\text{(partial) symmetry}}{\longleftrightarrow} \text{compact saturated.}$$

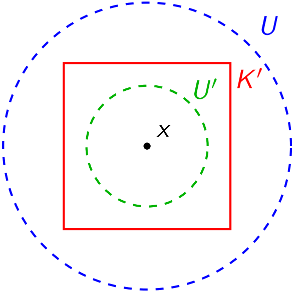
Definition

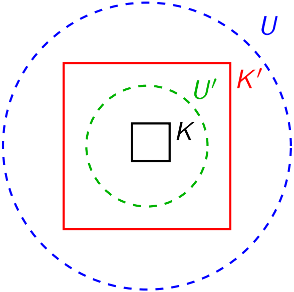
A topological space is *well-filtered* if for every open U and every codirected set $\mathcal{F}$ of compact saturated sets with $\bigcap \mathcal{F} \subseteq U$ there is $K \in \mathcal{F}$ such that $K \subseteq U$.

Hausdorff $\Rightarrow$ sober $\Rightarrow$ well-filtered.

Definition (Local compactness)

A space X is *locally compact* if, for every $x \in X$ and open $U \ni x$, there are an open U' and a compact K' such that $x \in U' \subseteq K' \subseteq U$.





Equivalently:

for every compact saturated K and open U with $K \subseteq U$, there is an open U' and a compact saturated K' such that $K \subseteq U' \subseteq K' \subseteq U$.

It's a symmetrical condition.

An important class:

$$\text{locally compact well-filtered } T_0 \text{ spaces}$$

(equivalently, locally compact sober spaces).

For example, they are categorically dual to locally compact frames (= continuous complete lattices).

These are almost symmetrical.

Non-symmetry: in locally compact well-filtered T_0 spaces,

- open sets are closed under finite unions, but
- compact saturated sets may not be closed under finite intersections.

Stably compact spaces: locally compact T_0 well-filtered spaces in which compact saturated sets are closed under finite intersections.

Examples:

- any compact Hausdorff space,
- any spectral space,
- $[0, 1]$ with the upper topology.

Stably compact spaces enjoy a complete symmetry between open subsets and compact saturated subsets.

For every stably compact space X there is another stably compact space X^∂, called the *de Groot dual* of X, with the same underlying set, and

- open of X^∂ = complement of compact saturated of X,
- compact saturated of X^∂ = complement of open of X.

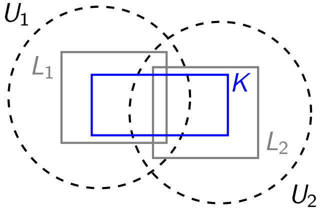
$$(X^\partial)^\partial = X.$$

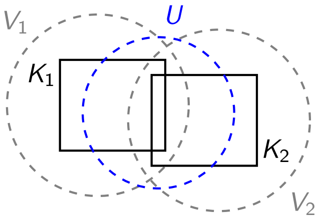
Example: the de Groot dual of $[0, 1]$ with the upper topology is $[0, 1]$ with the lower topology.

If all stably compact spaces enjoy a certain property, then they enjoy also the "symmetric" property, obtained by swapping open sets with compact saturated sets and reversing the inclusions.

Theorem (Wilker's conditions)

1. *Let X be a stably compact space (e.g.: compact Hausdorff space), K a compact saturated set, and U_1, U_2 open sets with $K \subseteq U_1 \cup U_2$. There are compact saturated sets L_1, L_2 such that $L_1 \subseteq U_1$, $L_2 \subseteq U_2$ and $K \subseteq L_1 \cup L_2$.*
2. *Let X be a stably compact space (e.g.: compact Hausdorff space), U an open set, and K_1, K_2 compact saturated sets with $K_1 \cap K_2 \subseteq U$. There are open sets V_1, V_2 such that $K_1 \subseteq V_1$, $K_2 \subseteq V_2$ and $V_1 \cap V_2 \subseteq U$.*





One can be deduced from the other one.

Theorem (Wilker's conditions)

1. *Let X be a locally compact well-filtered T_0 space, K a compact saturated set, and U_1, U_2 open sets with $K \subseteq U_1 \cup U_2$. There are compact saturated sets L_1, L_2 such that $L_1 \subseteq U_1$, $L_2 \subseteq U_2$ and $K \subseteq L_1 \cup L_2$.*
2. *Let X be a stably locally compact space, U an open set, and K_1, K_2 compact saturated sets with $K_1 \cap K_2 \subseteq U$. There are open sets V_1, V_2 such that $K_1 \subseteq V_1$, $K_2 \subseteq V_2$ and $V_1 \cap V_2 \subseteq U$.*

Stably locally compact := locally compact well-filtered T_0 + the set of compact saturated sets is closed under binary intersections.

We cannot deduce one from the other!

$$\varnothing \text{ is open} \leftrightsquigarrow X \text{ compact saturated.}$$

We need a generalization of topological space.

In Stone-like dualities, the interaction between open sets and compact saturated sets seems to be central:

1. every locally compact frame is spatial.
2. the spatial frames are precisely those with enough Scott-open filters (in bijection with compact saturated sets of the sober space).
3. In Stone's dualities for Boolean algebras and for bounded distributive lattices, to a space one associates the algebra of subsets that are both open and compact saturated.

The classes of spaces involved in dualities have strong signs of symmetry:

- Stone spaces (total symmetry between open and comp. saturated),
- spectral spaces (total symmetry),
- compact Hausdorff spaces (total symmetry),
- stably compact spaces (total symmetry),
- locally compact Hausdorff spaces (partial symmetry),
- locally compact sober spaces (partial symmetry),
- sober spaces (partial symmetry).

(The symmetry is reflected in the symmetry of logical calculus.)

Can we have a general result with total symmetry that covers these?

There are so many signs of symmetry between openness and compact saturatedness, especially in categorical dualities.

Sometimes the symmetry is perfect. But to cover more cases (e.g. sober spaces), we lose the perfect symmetry; only a partial symmetry remains.

There must be a setting with a complete symmetry and which is general enough to cover these cases.

Can we get a setting in which open and compact saturated sets are perfectly symmetrical notions and which is general enough to

1. cover the two Wilker's conditions, and
2. cover the main classes of spaces appearing in Stone-like dualities

?

A one-family solution

Open sets are closed under

1. finite intersections,
2. directed unions,
3. finite unions.

In well-filtered spaces, compact saturated sets are closed under

1. finite unions,
2. codirected intersections,

but not in general under finite intersections.

[Erné, 2007] considered sets equipped with a set of subsets closed under finite intersections and directed unions.

Definition (Directed-space)

A *directed-space* $(X, \mathcal{O})$ is a set X equipped with a set $\mathcal{O}$ of subsets closed under directed unions.

The elements of $\mathcal{O}$ are called the *open* subsets of X.

1. A set $S \subseteq X$ is *compact* if for every directed subset $\mathcal{I}$ of $\mathcal{O}$ with $S \subseteq \bigcup \mathcal{I}$ there is $U \in \mathcal{I}$ with $S \subseteq U$.
2. We say that S is *saturated* if it is the codirected intersection of $\{U \in \mathcal{O} \mid S \subseteq U\}$.

S is compact and saturated if and only if $\{U \in \mathcal{O} \mid S \subseteq U\}$ is a Scott-open filter of $\mathcal{O}$ with intersection S.

Given a directed-space $(X, \mathcal{O})$, the *de Groot dual* of $(X, \mathcal{O})$ is

$$(X, \{X \setminus K \mid K \text{ compact and saturated}\}).$$

De Groot duality is a well-defined involution on the class of directed-spaces satisfying

- local compactness
- well-filteredness
- every open is the directed union of the compact saturated sets in it.

This class contains all locally compact well-filtered topological spaces.

Theorem (Wilker's conditions)

Let X be a locally compact well-filtered directed-space in which every open set is the directed union of the compact saturated sets it contains.

1. *Suppose that open sets are closed under binary unions. Let K be a compact saturated set, and U_1, U_2 open sets with $K \subseteq U_1 \cup U_2$. There are compact saturated sets L_1, L_2 such that $L_1 \subseteq U_1$, $L_2 \subseteq U_2$ and $K \subseteq L_1 \cup L_2$.*
2. *Suppose that compact saturated sets are closed under binary intersections. Let U be an open set, and K_1, K_2 compact saturated sets with $K_1 \cap K_2 \subseteq U$. There are open sets V_1, V_2 such that $K_1 \subseteq V_1$, $K_2 \subseteq V_2$ and $V_1 \cap V_2 \subseteq U$.*

Any of the two can be deduced from the other one.

The topological case (for locally compact well-filtered T_0 spaces and stably locally compact spaces) is a special case.

However, outside local compactness...

Any class of directed-spaces on which de Groot duality is a well-defined involution will miss some Hausdorff topological spaces. Indeed, there are Hausdorff spaces which are not their own double de Groot dual.

E.g.: Arens-Fort space, Fortissimo spaces: Hausdorff non-discrete spaces where compact $\Leftrightarrow$ finite (and so their double de Groot dual is discrete).

But all Hausdorff spaces appear in the duality with spatial frames, so we don't want to lose them.

Solution:

open and compact saturated as non-interdefinable primitive notions.

A two-family solution

If X is a T_0 well-filtered space (e.g., any sober space, and in particular any Hausdorff space), then consider the structure

$$(X, \mathcal{K}, \mathcal{O})$$

where

- X is equipped with the specialization order ($x \leq y$ iff every open containing x contains y),
- $\mathcal{K} = \{\text{compact saturated sets}\}$,
- $\mathcal{O} = \{\text{open sets}\}$.

Definition

A *sober bi-directed-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}$ (the "*k-sets*") and $\mathcal{O}$ (the "*o-sets*") sets of upsets of X s.t.

1. ((Co)directedness)
   $\mathcal{K}$ is closed under codirected intersections and $\mathcal{O}$ under directed unions.
2. (Double compactness)
   - For every $K \in \mathcal{K}$ and directed $\mathcal{I} \subseteq \mathcal{O}$ with $K \subseteq \bigcup \mathcal{I}$, there is $U \in \mathcal{I}$ with $K \subseteq U$.
   - For every $U \in \mathcal{O}$ and codirected $\mathcal{F} \subseteq \mathcal{K}$ with $\bigcap \mathcal{F} \subseteq U$, there is $K \in \mathcal{F}$ with $K \subseteq U$.
3. (Principal upsets/downsets are compact/closed)
   For every $x \in X$, $\uparrow x \in \mathcal{K}$ and $X \setminus \downarrow x \in \mathcal{O}$.

These axioms are weak enough to capture all well-fiitered T_0 spaces, but strong enough to obtain a duality with certain pointfree structures.

Given a sober bi-directed-space $(X, \mathcal{K}, \mathcal{O})$, consider the triple

$$(\mathcal{K}, \mathcal{O}, \lhd)$$

where

- $\mathcal{K}$ is equipped with the inclusion order
- $\mathcal{O}$ is equipped with the inclusion order,
- $\lhd \colon \mathcal{K} \looparrowright \mathcal{O}$ is the inclusion relation.

Definition

A *bi-dcpo* is a triple $(\mathcal{K}, \mathcal{O}, \lhd)$ where $\mathcal{K}$ is a co-dpco, $\mathcal{O}$ is a dcpo, and $\lhd \colon \mathcal{K} \nrightarrow \mathcal{O}$ is a relation satisfying

1. (Double compactness)
   1.1 (Compactness of k-elements)
   For every $k \in \mathcal{K}$ and every directed subset I of $\mathcal{O}$ with $k \lhd \bigvee I$, there is $u \in I$ with $k \lhd u$.
   1.2 (Cocompactness of o-elements)
   For every $u \in \mathcal{O}$ and every codirected subset F of $\mathcal{K}$ with $\bigwedge F \lhd u$, there is $k \in F$ with $k \lhd u$.
2. (Weakening)
   $k' \leq k \lhd u \leq u'$ implies $k' \lhd u'$
3. (Extensionality)
   $\downarrow_{\mathcal{K}} u \subseteq \downarrow_{\mathcal{K}} v$ implies $u \leq v$, and $\uparrow_{\mathcal{O}} k \supseteq \uparrow_{\mathcal{O}} l$ implies $k \leq l$.

If $\mathcal{O}$ is a spatial frame (or any dcpo with enough Scott-open filters), then

$$(\{\text{Scott-open filters}\}, \mathcal{O}, \ni)$$

is a bi-dcpo.

Theorem

$$\begin{array}{ccc} \textit{Sober bi-directed-spaces} & \stackrel{1:1}{\longleftrightarrow} & \textit{distributive bi-dcpo.} \\ (X, \mathcal{K}, \mathcal{O}) & \longmapsto & (\mathcal{K}, \mathcal{O}, \subseteq) \end{array}$$

Distributivity := ... (a certain first-order condition)

Open sets and compact saturated sets are elements of the complete lattice of upsets. For every bi-dcpo $(\mathcal{K}, \mathcal{O}, \lhd)$ there is a canonical complete lattice L (its *concept lattice*), in which $\mathcal{K}$ and $\mathcal{O}$ embed.

Distributivity of $(\mathcal{K}, \mathcal{O}, \lhd) \iff$ distributivity of its concept lattice.

This correspondence restricts to the one between sober spaces, seen as

$$(\text{space}, \{\text{compact saturated sets}\}, \{\text{open sets}\}),$$

and spatial frames, seen as

$$(\text{frame}, \{\text{Scott open filters}\}, \text{reverse membership}).$$

$$\text{Sober bi-directed-spaces} \stackrel{1:1}{\longleftrightarrow} \text{distributive bi-dcpo.}$$

Both parts are "self-dual", also in a categorical sense.

Morphisms of sober bi-directed-spaces: "closed relations". "De Groot self-duality".

Morphisms of bi-dcpos: "adjoint pairs" (à la Chu). "Lawson self-duality".

Theorem

The category of sober bi-directed-spaces is equivalent (and dually equivalent) to the category of distributive bi-dcpos.

It embeds the duality between sober spaces and spatial frames into a duality with perfect symmetries.