The algebraic structure of spaces of integrable functions

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Based on:

Marco Abbadini.
Operations that preserve integrability, and truncated Riesz spaces.
Forum Mathematicum, 2020.

Algebraic approach to measure theory.

T. Kroupa, V. Marra. The two-sorted algebraic theory of states, and the universal states of MV-algebras. *Journal of Pure and Applied Algebra*, 2021.

Measure space:

$$(\underbrace{\Omega}_{\text{set}}, \underbrace{\mathcal{F}}_{\sigma\text{-algebra}}, \underbrace{\mu\colon \mathcal{F} \to [0,\infty]}_{\text{measure}})$$

E.g.:

$$(\mathbb{R}, \{\text{Lebesgue measurable sets}\}, \text{Lebesgue measure})$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A function $f \colon \Omega \to \mathbb{R}$ is called *integrable* if it is $\mathcal{F}$-measurable and $\int_\Omega |f| \, d\mu < \infty$.

$$\mathcal{L}^1(\mu) \coloneqq \{f \colon \Omega \to \mathbb{R} \mid f \text{ is integrable}\}.$$

T. Kroupa, V. Marra. The two-sorted algebraic theory of states, and the universal states of MV-algebras. *Journal of Pure and Applied Algebra*, 2021.

Idea: to see an integral

$$\int : \mathcal{L}^1(\mu) \to \mathbb{R}$$

as a two-sorted algebra, with

- $\mathcal{L}^1(\mu)$ the content of one sort (and an algebra of its own),
- $\mathbb{R}$ the content of the other sort (and an algebra of its own),
- $\int$ the interpretation of a unary function symbol between the two sorts.

What algebraic structure does $\mathcal{L}^1(\mu)$ have?

E.g.: it has at least a vector space structure, obtained via pointwise application of the vector space structure of $\mathbb{R}$.

What is the algebraic structure common to all $\mathcal{L}^1(\mu)$'s?

Example

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $f, g \colon \Omega \to \mathbb{R}$ integrable functions. Then

- $f + g$ is integrable,
- $f \cdot g$ may be non-integrable. (E.g.: $\Omega = (0, 1)$, $f(x) = g(x) = \frac{1}{x^{0.9}}$.)

The addition $+ \colon \mathbb{R}^2 \to \mathbb{R}$, applied pointwise, *preserves integrability*. Not the multiplication $\cdot \colon \mathbb{R}^2 \to \mathbb{R}$.

Preserving integrability $:=$ returning integrable functions when applied to integrable functions.

Definition

A function $\tau \colon \mathbb{R}^\kappa \to \mathbb{R}$ (with κ a cardinal) *preserves integrability* if for every measure space $(\Omega, \mathcal{F}, \mu)$ and all $(f_i \in \mathcal{L}^1(\mu))_{i \in \kappa}$ we have $\tau((f_i)_{i \in \kappa}) \in \mathcal{L}^1(\mu)$.

Question 1

Clone on $\mathbb{R}$ of functions $\mathbb{R}^\kappa \to \mathbb{R}$ that preserve integrability?

Question 2

Simple set of generators?

Question 3

Axiomatization of the variety generated by $\mathbb{R}$?

For every measure space $(\Omega, \mathcal{F}, \mu)$, $\mathcal{L}^1(\mu)$ belongs to this variety.

Question 1:
which functions $\mathbb{R}^{\kappa} \to \mathbb{R}$ preserve integrability?

Theorem (A., 2020)

For every $n \in \mathbb{N}$, a function $\tau \colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability if and only if it is Borel measurable and sublinear.

Borel measurable := the preimage of a Borel measurable set is Borel measurable.

Sublinear := there are positive real numbers $\lambda_1, \ldots, \lambda_n$ such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$|\tau(\mathbf{x})| \leq \sum_{i=1}^{n} \lambda_i |x_i|.$$

i.e., τ...

- is at most linear in a neighbourhood of ∞;
- is at most linear in a neighbourhood of 0;
- is bounded on bounded sets.

Example

All linear operations are Borel measurable and sublinear, and hence preserve integrability:

- The addition $+\colon \mathbb{R}^2 \to \mathbb{R}$;
- For every $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda \cdot - \colon \mathbb{R} \to \mathbb{R}$ by λ;
- The constant $0\colon \mathbb{R}^0 \to \mathbb{R}$.

I.e.: every $\mathcal{L}^1(\mu)$ is a vector space: all linear operations are well-defined (via pointwise application).

Example

The square function $(-)^2\colon \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable but not sublinear (at infinity it grows more than linearly).

$$\downarrow$$

The square function does not preserve integrability, i.e.,

$$f \text{ integrable} \not\Rightarrow f^2 \text{ integrable.}$$

Counterexample: take a big f on a small measure space.

Example

The square root function $\sqrt{|-|}\colon \mathbb{R} \to \mathbb{R}$ is Borel measurable but not sublinear, because no linear function bounds it close to zero.

$$\downarrow$$

The square root function does not preserve integrability, i.e.,

$$f \text{ integrable} \not\Rightarrow \sqrt{|f|} \text{ integrable.}$$

Counterexample: Take a small f on a big measure space.

Example

max, min: $\mathbb{R}^2 \to \mathbb{R}$ are Borel measurable and sublinear:

$$|\max\{x, y\}| \leq |x| + |y|.$$

$$\downarrow$$

max and min preserve integrability, i.e.,

$$f, g \text{ integrable} \Rightarrow \sup\{f, g\}, \inf\{f, g\} \text{ integrable.}$$

Essentially: every $\mathcal{L}^1(\mu)$ has a lattice structure induced by $\mathbb{R}$.

Example

The constant $1\colon \mathbb{R}^0 \to \mathbb{R}$ is Borel measurable but not sublinear (at $\mathbf{0}$ it is not 0.)

$$\downarrow$$

1 does not preserve integrability. E.g.: the constant function 1 on $\mathbb{R}$ is not integrable.

For arbitrary arity:

Theorem (A., 2020)

For every cardinal κ, a function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability if and only if τ is cylinder measurable and sublinear.

Cylinder (or *product*) σ-algebra on $\mathbb{R}^\kappa :=$ the smallest σ-algebra that makes all projections measurable $=$ the σ-algebra generated by the sets

$$\prod_{i \in \kappa} \begin{cases} A & \text{if } i = i_0; \\ \mathbb{R} & \text{if } i \neq i_0; \end{cases}$$

for $i_0 \in \kappa$ and $A \subseteq \mathbb{R}$ Borel.

For κ finite or countable, cylinder σ-algebra on $\mathbb{R}^\kappa =$ Borel σ-algebra on $\mathbb{R}^\kappa$.

Sublinear $:=$ there are distinct $i_1, \dots i_k \in \kappa$ s.t., for every $\mathbf{x} \in \mathbb{R}^\kappa$,

$$|\tau(\mathbf{x})| \leq \sum_{j=1}^{k} \lambda_i |x_{i_j}|.$$

Example

While the countable supremum is not well-defined on $\mathbb{R}$, the truncated countable supremum

$$(y, x_1, x_2, \dots) \mapsto \sup_{i=1}^{\infty} \min\{y, x_i\}$$

is well-defined. It is cylinder (= Borel) measurable and sublinear:

$$|\sup_{i=1}^{\infty} \min\{y, x_i\}| \leq |y| + |x_1|.$$

$$\downarrow$$

It preserves integrability, i.e.:

$$g, f_1, f_2, \dots \text{ integrable} \Rightarrow \sup_{i=1}^{\infty} \inf\{g, f_i\} \text{ integrable.}$$

Definition

A function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ (with κ a cardinal) *preserves integrability over finite measure spaces* if for every measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$ and all $(f_i \in \mathcal{L}^1(\mu))_{i\in\kappa}$ we have $\tau((f_i)_{i\in\kappa}) \in \mathcal{L}^1(\mu)$.

Theorem (A., 2020)

For every cardinal κ, a function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability over finite measure spaces if and only if it is cylinder measurable and subaffine.

Subaffine := there are positive real numbers $k, \lambda_1, \ldots, \lambda_n$ such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$|\tau(\mathbf{x})| \leq k + \sum_{i=1}^{n} \lambda_i |x_i|.$$

i.e., τ...

- is at most linear in a neighbourhood of ∞;
- ~~is at most linear in a neighbourhood of 0;~~
- is bounded on bounded sets.

The function $\sqrt{|-|}\colon \mathbb{R} \to \mathbb{R}$ is measurable, (not sublinear, but) subaffine. Hence, it preserves integrability over finite measure spaces.

The constant function $1\colon \mathbb{R}^0 \to \mathbb{R}$ preserves integrability over finite measure spaces.

Question 2:
set of generators?

Theorem (A., 2020)

The infinitary clone on $\mathbb{R}$ of functions that preserve integrability (= measurable + sublinear) is generated by

- *"linear" operations:*
  - 0;
  - $+$;
  - *for $\lambda \in \mathbb{R}$, the scalar multiplication by λ, i.e. $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda \cdot x$;*
- *the truncation by 1, i.e. $\mathbb{R} \to \mathbb{R}$, $x \longmapsto x \wedge 1$;*
- *the truncated countable supremum*

$$(y, x_1, x_2, \dots) \mapsto \sup_{n=1}^{\infty}\{\min\{y, x_n\}\}.$$

Theorem (A., 2020)

The infinitary clone on $\mathbb{R}$ of functions that preserve integrability over finite measure spaces (= measurable + subaffine) is generated by

- *"affine" operations:*
  - 0;
  - $+$;
  - *for $\lambda \in \mathbb{R}$, the scalar multiplication by λ, i.e. $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda \cdot x$;*
  - 1;
- *the truncated countable supremum*

$$(y, x_1, x_2, \dots) \mapsto \sup_{n=1}^{\infty}\{\min\{y, x_n\}\}.$$

Question 3:
axiomatization?

Recall: a function $\mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability over *finite* measure spaces if and only if it is measurable and subaffine.

Theorem (A., 2020)

The variety generated by the clone of measurable subaffine functions on $\mathbb{R}$ is the class of Dedekind σ-complete vector lattice with a weak unit.

Vector lattice (a.k.a. *Riesz space*) $:=$ vector space $+$ lattice order $+$ compatibility conditions.

Dedekind σ-complete $:=$ every countable subset with an upper bound has a supremum. (Equivalently, for vector lattices: every countable subset with a lower bound has an infimum.)

Weak unit $:=$ element $1 \geq 0$ such that $x \wedge 1 = 0$ implies $x = 0$.

The structure of a Dedekind σ-complete vector lattice with a weak unit is the richest algebraic structure shared by all $\mathcal{L}^1(\mu)$ with μ finite (and that uses only pointwise applications of functions $\mathbb{R}^\kappa \to \mathbb{R}$).

Recall: a function $\mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability if and only if it is measurable and sublinear.

Theorem (A., 2020)

The variety generated by the clone of measurable sublinear functions on $\mathbb{R}$ *is the class of Dedekind* σ*-complete truncated vector lattices.*

Truncated [Ball, 2014] $:=$ ... $=$ with a unary endofunction behaving like a meet with a weak unit $a \mapsto a \wedge 1$.

The structure of a Dedekind σ-complete truncated vector lattice is the richest algebraic structure shared by all $\mathcal{L}^1(\mu)$'s (and that uses only pointwise applications of functions $\mathbb{R}^\kappa \to \mathbb{R}$).

Theorem (A., 2020)

Dedekind σ-complete vector lattice with a weak unit form a variety of algebras, generated by $\mathbb{R}$.

$$\{\text{Ded. } \sigma\text{-compl. vect. latt. w. weak unit}\} = \mathbb{HSP}(\mathbb{R})$$

Theorem (A., 2020)

Dedekind σ-complete vector lattice with a weak unit form a variety of algebras, generated by $\mathbb{R}$.

$$\{\text{Ded. } \sigma\text{-compl. vect. latt. w. weak unit}\} = \mathbb{HSP}(\mathbb{R}) = \mathbb{ISP}_{\sigma\text{-reduced}}(\mathbb{R}).$$

In fact, every Dedekind σ-complete vector lattice with a weak unit is a subalgebra of a power of $\mathbb{R}$ modulo a σ-ideal.

($\approx$ Loomis-Sikorski theorem.)

Analogous results hold for Dedeking σ-complete truncated vector lattices.

Theorem (A., 2021)

The free Dedekind σ-complete vector lattice with a weak unit over a set I (exists, and) is

$$\left\{\textit{measurable and subaffine } \mathbb{R}^I \to \mathbb{R}\right\}.$$

Theorem (A., 2021)

The free Dedekind σ-complete truncated vector lattice over a set I (exists, and) is

$$\left\{\textit{measurable and sublinear } \mathbb{R}^I \to \mathbb{R}\right\}.$$

To sum up

$\mathcal{L}^1(\mu)$'s: Dedekind σ-complete truncated vector lattice.

Finite $\mathcal{L}^1(\mu)$'s: Dedekind σ-complete vector lattice with a weak unit.

Free Dedekind σ-complete truncated vector lattice over I:

$$\left\{\text{measurable and sublinear } \mathbb{R}^I \to \mathbb{R}\right\}.$$

Free Dedekind σ-complete vector lattice with a weak unit over I:

$$\left\{\text{measurable and subaffine } \mathbb{R}^I \to \mathbb{R}\right\}.$$

$\mathcal{L}^1(\mu)$'s: Dedekind σ-complete truncated vector lattice.

Finite $\mathcal{L}^1(\mu)$'s: Dedekind σ-complete vector lattice with a weak unit.

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Thank you!

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