The unification type of Łukasiewicz logic with a bounded number of variables

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25 June 2025

Based on:

Marco Abbadini, Luca Spada.
Łukasiewicz unification with finitely many variables.
arXiv:2504.19011.

Let E be a set of equations in a functional signature (in our case, $E =$ the set of equations defining MV-algebras).

An *E-unification problem* is a pair (X, S) where X is a finite subset of Var and $S = \{s_1 \approx t_1, \dots, s_k \approx t_k\}$ is a finite set of equations with variables in X.

Example

$$x \vee y \vee \neg x \vee \neg y \approx 1. \tag{$\mathfrak{B}$}$$

An *E-unifier for (X, S) (with variables in a finite subset Y of Var)* is a substitution $\sigma \colon X \to \mathrm{Terms}(Y)$ such that, for all $i \in \{1, \dots, k\}$,

$$s_i[\sigma(x)/x,\ x \in X] \approx_E t_i[\sigma(x)/x,\ x \in X]$$

Example

A unifier for

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

with variables in $\{y\}$ is

$$x \mapsto 1, \; y \mapsto y,$$

because

$$1 \vee y \vee \neg 1 \vee \neg y \approx 1.$$

Example

$1 \approx 0$ has no unifier.

Example

$x \approx \neg x$ has no unifier.

Composing an E-unifier σ with a substitution gives still an E-unifier, which is "less general" than σ.

Example

Composing the unifier $x \mapsto 1$, $y \mapsto y$ of

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

with the substitution $y \mapsto y \oplus y$ gives a unifier (with variables in $\{z\}$)

$$x \mapsto 1, \; y \mapsto y \oplus y$$

because

$$1 \vee (y \oplus y) \vee \neg 1 \vee \neg(y \oplus y) \approx 1. \tag{$\mathfrak{B}$}$$

A unifier $\sigma\colon X \to \mathrm{Terms}(Y)$ is *more general* than $\sigma'\colon X \to \mathrm{Terms}(Y')$ (with respect to E) if there is a substitution $\theta\colon Y \to \mathrm{Terms}(Y')$ such that, for all $x \in X$,

$$\sigma'(x) \approx_E (\sigma(x))[\theta(y)/y,\, y \in Y].$$

This relation defines a preorder on the set of unifiers.

Example

Is there a unifier for

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

that is strictly more general than

$$x \mapsto 1,\ y \mapsto y \tag{U1}$$

?

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?

Yes, for example, with variables in $\{x, y\}$:

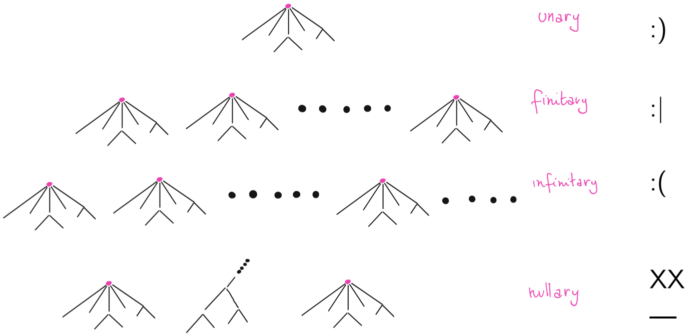
$$x \mapsto x \oplus y,\ y \mapsto x \odot y. \tag{U2}$$

It is a unifier:

$$(x \oplus y) \vee (x \odot y) \vee \neg(x \oplus y) \vee \neg(x \odot y) \approx 1.$$

(U2) is more general than (U1): $(x \mapsto 1,\ y \mapsto y) \circ \text{(U2)} = \text{(U1)}$. Not vice versa.

If a unification problem has a unifier, four situations may arise.



The unification type of E is the worst unification type occurring among unifiable unification problems.

The unification type of Łukasiewicz logic is the worst: **nullary** [Marra, Spada, 2013]. The problem

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

has nullary unification type.

Marra and Spada exhibited a chain (ordered as ω) of unifiers of strictly increasing generality such that every unifier of ($\mathfrak{B}$) is below some element in the chain.

They proved it via duality.

The unifiers in the chain are in a strictly increasing number of variables.

(Marra, Spada, 2013) Does the unification type of

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

and in general of Łukasiewicz logic improves by restricting to a finite number the number of available variables?

The unification type of Łukasiewicz logic restricted to 0 variables is unary.

Theorem (Marra, Spada, 2013)

The unification type of Łukasiewicz logic restricted to 1 *variable is finitary.*

Conjecture (Marra, Spada, 2013)

For all $n \geqslant 2$, the unification type of Łukasiewicz logic restricted to n variables is nullary.

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Theorem (A., Spada)

Correct.

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The unification type of Łukasiewicz logic restricted to 1 *variable is finitary.*

Conjecture (Marra, Spada, 2013)

For all $n \geqslant 2$, the unification type of Łukasiewicz logic restricted to n variables is nullary.

Theorem (A., Spada)

Correct.

Reason: the problem

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

is still NUTs (of nullary unification types).

Theorem (A., Spada)

For every $n \geqslant 2$, the preordered set of unifiers for

$$x \vee y \vee \neg x \vee \neg y \approx 1 \tag{$\mathfrak{B}$}$$

with at most n variables has nullary unification type.

We prove that every unifier in at most n variables more general than

$$x \mapsto 1, \; y \mapsto y$$

is strictly less general than some unifier in at most n variables.

The proof is via duality.

The variables in the problem and the variables in the solution are pretty much unrelated.

Question

What is the unification type of Łukasiewicz logic restricted to $m \in \mathbb{N}$ variables for the problem and $n \in \mathbb{N}$ variables for the solution?

variables for the solution

variables for the problem

	0	1	2	3	4	5	...
0	unary						...
1		finitary					...
2							...
3							...
4							...
5							...
...	...	...	...	...	...	...	...

Marra & Spada, 2013.

A. & Spada.

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...	...	...	...	...	...	...	...

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1	finitary	finitary					...
2	finitary		nullary				...
3	finitary			nullary			...
4	finitary				nullary		...
5	finitary					nullary	...
...	...	...	...	...	...	...	...

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3	finitary		nullary	nullary	nullary	nullary	...
4	finitary		nullary	nullary	nullary	nullary	...
5	finitary		nullary	nullary	nullary	nullary	...
...	...	...	...	...	...	...	...

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	0	1	2	3	4	5	...
0	unary	unary	unary	unary	unary	unary	...
1	finitary	finitary	finitary	finitary	finitary	finitary	...
2	finitary	(open)	nullary	nullary	nullary	nullary	...
3	finitary	(open)	nullary	nullary	nullary	nullary	...
4	finitary	(open)	nullary	nullary	nullary	nullary	...
5	finitary	(open)	nullary	nullary	nullary	nullary	...
...	...	...	...	...	...	...	...

Marra & Spada, 2013.

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Theorem (See Marra, Spada, 2013)

The following categories are dually equivalent.

1. *The category of finitely presented MV-algebras and homomorphisms between them.*
2. *The category of rational polyhedra (i.e., finite unions of polytopes with rational vertices) and $\mathbb{Z}$-maps ($:=$ piecewise linear maps with integer coefficients) between them.*

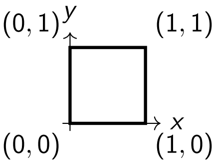
$$\text{Free MV-algebra on } n \in \mathbb{N} \text{ generators}$$
$$\updownarrow$$
$$[0,1]^n.$$

(The dual of) a *unification problem* consists of a natural number $m \in \mathbb{N}$ and a rational polyhedron $P \subseteq [0,1]^m$.

The dual of

$$x \vee y \vee \neg x \vee \neg y \approx 1. \tag{$\mathfrak{B}$}$$

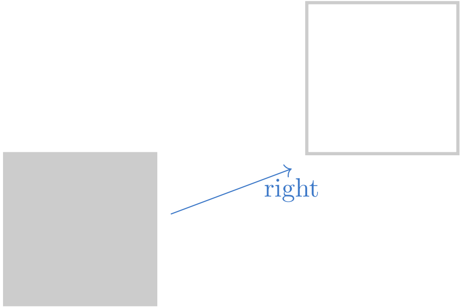
is the border of the square



(The dual of) a *unifier for P (with n variables)* is a $\mathbb{Z}$-map $[0,1]^n \to P$.

A unifier $f'\colon [0,1]^{n'} \to P$ is *more general* than a unifier $f\colon [0,1]^n \to P$ if there is a $\mathbb{Z}$-map $h\colon [0,1]^n \to [0,1]^{n'}$ such that the following diagram commutes.

$$\begin{tikzcd}
{[0,1]^{n'}} \arrow[r, "f'"] & B \\
{[0,1]^n} \arrow[u, dashed, "h"] \arrow[ur, "f"'] &
\end{tikzcd}$$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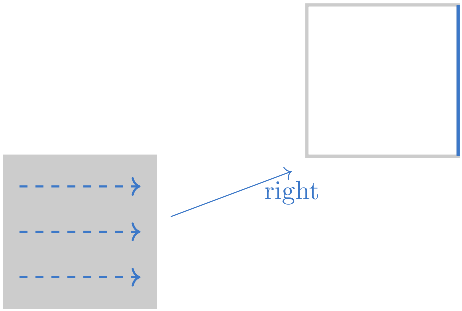
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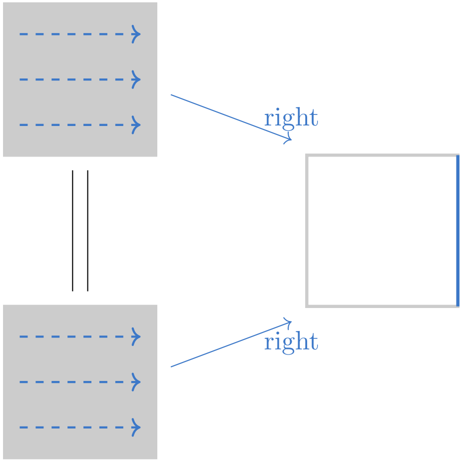
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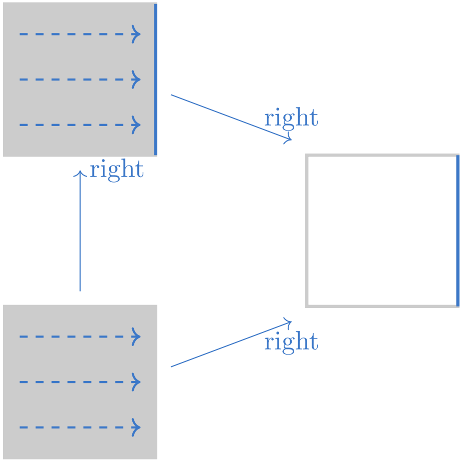
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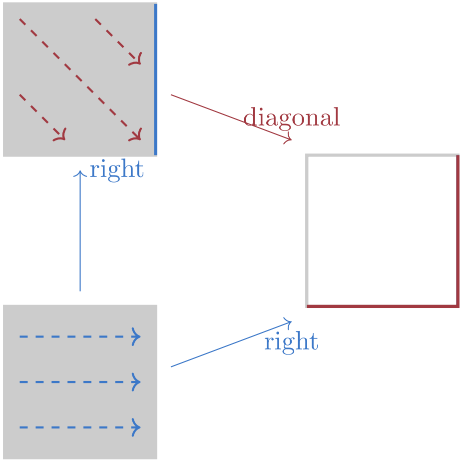
diagonal
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...	...	...	...	...	...	...	...

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Thank you