An algebraic version of Herbrand's theorem

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Based on a joint work with Francesca Guffanti:

Freely adding one layer of quantifiers to a Boolean doctrine. *arXiv*.

Universal theory: universal closures of quantifier-free formulas.

E.g.: theory of partial orders.

1. (Reflexivity) $\forall x\,(x \leq x)$,
2. (Transitivity) $\forall x\,\forall y\,((x \leq y \leq z) \to x \leq z)$
3. (Anti-symmetry) $\forall x\,\forall y\,((x \leq y) \wedge (y \leq x) \to x = y)$

Also, the theory of partial orders with min and max (as constants).

4. $\forall x\,(\min \leq x)$,
5. $\forall x\,(x \leq \max)$.

Herbrand's theorem for $\exists$-statements, 1930

Let $\mathcal{T}$ be a universal theory in a language with at least a constant symbol. For $x \in \mathrm{Var}$ and $\alpha(x)$ quantifier-free, $\mathcal{T}$ proves $\exists x\,\alpha(x)$ if and only if there are term-definable constants (i.e., ground terms) $c_1, \dots, c_n$ such that $\mathcal{T}$ proves $\alpha[{}^{c_1}/_{x}] \vee \dots \vee \alpha[{}^{c_n}/_{x}]$.

Examples:

- ($\Leftarrow$) The theory $\mathcal{T}$ of partial orders with min and max proves

$$\exists x\,(\max \leq x)$$

because $\mathcal{T}$ proves

$$\max \leq \max.$$

- ($\Rightarrow$) The theory $\mathcal{T}$ of partial orders with min and max does not prove

$$\exists x\,(\min \lneq x \lneq \max)$$

because $\mathcal{T}$ does not prove any of the following two

$$\min \lneq \min \lneq \max \qquad\qquad \min \lneq \max \lneq \max,$$

as well as any disjunction made up of them.

Herbrand's theorem

Let $\mathcal{T}$ be a universal theory in a language with at least a constant symbol. For $x \in \mathrm{Var}$ and $\alpha(x)$ quantifier-free, $\mathcal{T}$ proves $\exists x\, \alpha(x)$ if and only if there are term-definable constants $c_1, \dots, c_n$ such that $\mathcal{T}$ proves $\alpha[^{c_1}/_x] \vee \cdots \vee \alpha[^{c_n}/_x]$.

The hypothesis that there is at least a constant symbol cannot be removed: $\vdash \exists x\,(x = x)$, but no witnessing constants.

However, the hypothesis can be removed if we either

1. replace "term-definable constants" by "terms" (so that also variables can be used),
2. use a version of classical first-order logic whose semantics admits empty structures.

Boolean algebras : Classical propositional logic

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First-order Boolean doctrines [Lawvere, '60s] : Classical first-order logic.

Fix a functional language $\mathcal{F}$.

Roughly speaking: first-order Boolean doctrines over $\mathcal{F}$ are the algebras of the form

$$(\mathcal{F}, \mathcal{R})\text{-Formulas}/{\equiv_{\mathcal{T}}}$$

for $\mathcal{R}$ ranging among all sets of relation symbols, and $\mathcal{T}$ among all theories in the first-order language $(\mathcal{F}, \mathcal{R})$.

Usually, first-order Boolean doctrines are defined as certain functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{BoolAlg}$, with C taking the place of $\mathcal{F}$.

We avoid this categorical phrasing in this talk.

A *first-order Boolean doctrine* over a functional language $\mathcal{F}$ consists of

1. **a family of Boolean algebras** $(\mathbf{P}(X))_{X \subseteq_{\text{fin}} \mathrm{Var}}$ (here, $\mathbf{P}(X)$ models the set of equivalence classes of formulas with free variables in X),
2. **substitutions**: for $X, Y \subseteq_{\text{fin}} \mathrm{Var}$ and for a map $\sigma \colon X \to \mathrm{Term}(Y)$, we have a Boolean homomorphism $\mathbf{P}_\sigma \colon \mathbf{P}(X) \to \mathbf{P}(Y)$, modelling $\alpha \mapsto \alpha\left[\left({}^{\sigma(x)}/_{x}\right)_{x \in X}\right]$,
3. **quantifiers**: for $X \subseteq_{\text{fin}} \mathrm{Var}$, $y \in \mathrm{Var} \setminus X$ two maps $\exists, \forall \colon \mathbf{P}(X \cup \{y\}) \to \mathbf{P}(X)$, modelling $\alpha \mapsto \exists y\, \alpha$ and $\alpha \mapsto \forall y\, \alpha$,

satisfying certain axioms (functoriality of substitutions, quantifiers are adjoint to adding dummy variables, quantifiers commute with substitutions).

We avoid equality, for simplicity.

First-order Boolean doctrines are the algebras of classical first-order logic.

Disclaimer: the "classical first-order logic" meant here is the one whose semantics allows the usage of empty structures.

If empty structures are undesired, one can restrict to the first-order Boolean doctrines satisfying "$\exists x \top = \top$".

The algebras of quantifier-free formulas modulo a universal theory $\rightsquigarrow$ **Boolean doctrines**.

These are defined as **first-order Boolean doctrines**, but without quantifiers: a family of Boolean algebras, linked by substitutions (Boolean homomorphisms), satisfying functoriality.

First-order Boolean algebras over a fixed functional language $\mathcal{F}$ form a class of many-sorted algebras (one sort for each finite set of variables) definable by equations (i.e. a *variety of algebras*).

Varieties of algebras admit all free algebras. The forgetful functor from the category of **first-order Boolean doctrines** over $\mathcal{F}$ to the category of **Boolean doctrines** over $\mathcal{F}$ has a left adjoint: it maps a Boolean doctrine $\mathbf{P}$ to a first-order Boolean doctrine $\mathbf{P}^{\forall\exists}$, called the *quantifier completion* of $\mathbf{P}$, which "freely adds all quantifiers" to $\mathbf{P}$.

We have injections for each $X \subseteq_{\mathrm{fin}} \mathrm{Var}$

$$\mathbf{P}(X) \hookrightarrow \mathbf{P}^{\forall\exists}(X).$$

Roughly speaking: the set of quantifier-free formulas with variables in X is a subset of the set of first-order formulas with variables in X.

Herbrand's theorem

Let $\mathcal{T}$ be a universal theory in a language with at least a constant symbol. For $x \in \mathrm{Var}$ and $\alpha(x)$ quantifier-free, $\mathcal{T}$ proves $\exists x\, \alpha(x)$ if and only if there are term-definable constants $c_1, \dots, c_n$ such that $\mathcal{T}$ proves $\alpha[{}^{c_1}/_x] \vee \cdots \vee \alpha[{}^{c_n}/_x]$.

Algebraic version:

Theorem (A., Guffanti)

Let $\mathbf{P}$ be a Boolean doctrine over a functional signature. For $x \in \mathrm{Var}$ and $\alpha \in \mathbf{P}(\{x\})$,

$$\exists x\, \alpha = \top \quad \textit{in } \mathbf{P}^{\forall\exists}(\varnothing)$$

if and only if there are term-definable constants $c_1, \dots, c_n$ such that

$$\alpha[{}^{c_1}/_x] \vee \cdots \vee \alpha[{}^{c_n}/_x] = \top \quad \textit{in } \mathbf{P}(\varnothing).$$

No need for the existence of constants in $\mathcal{F}$. (Since $\varnothing$ is allowed.)

Herbrand's theorem

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$$\alpha[{}^{c_1}/_{x}] \vee \cdots \vee \alpha[{}^{c_n}/_{x}] = \top \quad \textit{in } \mathbf{P}(\varnothing).$$

$\mathcal{T}$ cannot be an arbitrary theory $\rightsquigarrow$ we need freeness of $\mathbf{P}^{\forall\exists}$ over $\mathbf{P}$.

We actually proved it in the setting of first-order Boolean doctrines as certain functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$.

In this way, we cover also *many-sorted* classical first-order logic.

The proof uses **models**. To produce them, we use a proof similar to Henkin's proof of Gödel's completeness theorem (using the axiom of choice).

Freely adding one layer of quantifiers to a Boolean doctrine. *arXiv*.

Thank you!