The algebraic structure of spaces of integrable functions

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Based on:

Marco Abbadini.
Operations that preserve integrability, and truncated Riesz spaces.
Forum Mathematicum, 2020.

Algebraic approach to measure theory.

T. Kroupa, V. Marra. Generalised states: a multi-sorted algebraic approach to probability. *Soft Computing*, 2017.

T. Kroupa, V. Marra. The two-sorted algebraic theory of states, and the universal states of MV-algebras. *Journal of Pure and Applied Algebra*, 2021.

Measure space:

$$(\underbrace{\Omega}_{\text{set}}, \underbrace{\mathcal{F}}_{\sigma\text{-algebra}}, \underbrace{\mu\colon \mathcal{F} \to [0,\infty]}_{\text{measure}})$$

E.g.:

$$(\mathbb{R}, \text{ Borel } \sigma\text{-algebra}, \text{ Lebesgue measure})$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A function $f\colon \Omega \to \mathbb{R}$ is called *integrable* if it is measurable (with respect to $\mathcal{F}$ and the Borel σ-algebra on $\mathbb{R}$) and $\int_\Omega |f| \, d\mu < \infty$.

$$\mathcal{L}^1(\mu) \coloneqq \{f\colon \Omega \to \mathbb{R} \mid f \text{ is integrable}\}.$$

T. Kroupa, V. Marra. The two-sorted algebraic theory of states, and the universal states of MV-algebras. *Journal of Pure and Applied Algebra*, 2021.

Idea: to see an integral

$$\int : \mathcal{L}^1(\mu) \to \mathbb{R}$$

as a two-sorted algebra, with

- $\mathcal{L}^1(\mu)$ the content of one sort (and an algebra of its own),
- $\mathbb{R}$ the content of the other sort (and an algebra of its own),
- $\int$ the interpretation of a unary function symbol between the two sorts.

What algebraic structure does $\mathcal{L}^1(\mu)$ have?

E.g.: it has at least a vector space structure, obtained via pointwise application of the vector space structure of $\mathbb{R}$.

What is the algebraic structure common to all $\mathcal{L}^1(\mu)$'s?

Example

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $f, g \colon \Omega \to \mathbb{R}$ integrable functions. Then

- $f + g$ is integrable,
- $f \cdot g$ may be non-integrable. (E.g.: $\Omega = (0,1)$, $f(x) = g(x) = \frac{1}{x^{0.9}}$.)

The addition $+\colon \mathbb{R}^2 \to \mathbb{R}$, applied pointwise, *preserves integrability*. Not the multiplication $\cdot\colon \mathbb{R}^2 \to \mathbb{R}$.

Preserving integrability $:=$ returning integrable functions when applied to integrable functions.

Definition

A function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ (with κ a cardinal) *preserves integrability* if for every measure space $(\Omega, \mathcal{F}, \mu)$ and all $(f_i \in \mathcal{L}^1(\mu))_{i\in\kappa}$ we have $\tau((f_i)_{i\in\kappa}) \in \mathcal{L}^1(\mu)$.

Question 1

Clone on $\mathbb{R}$ of functions $\mathbb{R}^\kappa \to \mathbb{R}$ that preserve integrability?

Question 2

Simple set of generators?

Question 3

Axiomatization of the variety generated by $\mathbb{R}$?

For every measure space $(\Omega, \mathcal{F}, \mu)$, $\mathcal{L}^1(\mu)$ belongs to this variety.

Question 1:

which functions $\mathbb{R}^\kappa \to \mathbb{R}$ preserve integrability?

Theorem (A., 2020)

For every $n \in \mathbb{N}$, a function $\tau\colon \mathbb{R}^n \to \mathbb{R}$ preserves integrability if and only if it is Borel measurable and sublinear.

Borel measurable := the preimage of a Borel measurable set is Borel measurable.

Sublinear := there are positive real numbers $\lambda_1, \dots, \lambda_n$ such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$|\tau(\mathbf{x})| \leq \sum_{i=1}^{n} \lambda_i |x_i|.$$

i.e., τ...

- is at most linear in a neighbourhood of ∞;
- is at most linear in a neighbourhood of 0;
- is bounded on bounded sets.

Example

All linear operations are Borel measurable and sublinear, and hence preserve integrability:

- The addition $+\colon \mathbb{R}^2 \to \mathbb{R}$;
- For every $\lambda \in \mathbb{R}$, the scalar multiplication $\lambda \cdot -\colon \mathbb{R} \to \mathbb{R}$ by λ;
- The constant $0\colon \mathbb{R}^0 \to \mathbb{R}$.

I.e.: every $\mathcal{L}^1(\mu)$ is a vector space: all linear operations are well-defined (via pointwise application).

Example

The square function $(-)^2\colon \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable but not sublinear (at infinity it grows more than linearly).

$$\downarrow$$

The square function does not preserve integrability, i.e.,

$$f \text{ integrable} \not\Rightarrow f^2 \text{ integrable.}$$

Counterexample: take a big f on a small measure space.

Example

The square root function $\sqrt{|-|}\colon \mathbb{R} \to \mathbb{R}$ is Borel measurable but not sublinear, because no linear function bounds it close to zero.

$$\downarrow$$

The square root function does not preserve integrability, i.e.,

$$f \text{ integrable} \not\Rightarrow \sqrt{|f|} \text{ integrable.}$$

Counterexample: Take a small f on a big measure space.

Example

max, min: $\mathbb{R}^2 \to \mathbb{R}$ are Borel measurable and sublinear:

$$|\max\{x, y\}| \leq |x| + |y|.$$

$$\downarrow$$

max and min preserve integrability, i.e.,

$$f, g \text{ integrable} \Rightarrow \sup\{f, g\}, \inf\{f, g\} \text{ integrable.}$$

Essentially: every $\mathcal{L}^1(\mu)$ has a lattice structure induced by $\mathbb{R}$.

Example

The constant $1\colon \mathbb{R}^0 \to \mathbb{R}$ is Borel measurable but not sublinear (at $\mathbf{0}$ it is not 0.)

$$\downarrow$$

1 does not preserve integrability. E.g.: the constant function 1 on $\mathbb{R}$ is not integrable.

For arbitrary arity:

Theorem (A., 2020)

For every cardinal κ, a function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability if and only if τ is measurable (with respect to the product σ-algebra of $\mathbb{R}^\kappa$) and sublinear.

Product (or *cylinder*) σ-algebra on $\mathbb{R}^\kappa$:= smallest σ-algebra that makes all projections $\mathbb{R}^\kappa \to \mathbb{R}$ measurable = σ-algebra generated by the sets

$$\prod_{i\in\kappa} \begin{cases} A & \text{if } i = i_0; \\ \mathbb{R} & \text{if } i \neq i_0; \end{cases}$$

for $i_0 \in \kappa$ and $A \subseteq \mathbb{R}$ Borel.

For κ finite or countable, product σ-alg. on $\mathbb{R}^\kappa$ = Borel σ-alg. on $\mathbb{R}^\kappa$.

Sublinear := there are distinct $i_1, \dots i_k \in \kappa$ s.t., for every $\mathbf{x} \in \mathbb{R}^\kappa$,

$$|\tau(\mathbf{x})| \leq \sum_{j=1}^{k} \lambda_i |x_{i_j}|.$$

Example

While the countable supremum is not well-defined on $\mathbb{R}$, the truncated countable supremum

$$(y, x_1, x_2, \dots) \mapsto \sup_{i=1}^{\infty} \min\{y, x_i\}$$

is well-defined. It is measurable wrt to the product ($=$ Borel) σ-algebra and it is sublinear:

$$|\sup_{i=1}^{\infty} \min\{y, x_i\}| \leq |y| + |x_1|.$$

$$\downarrow$$

It preserves integrability, i.e.:

$$g, f_1, f_2, \dots \text{ integrable} \Rightarrow \sup_{i=1}^{\infty} \inf\{g, f_i\} \text{ integrable.}$$

Definition

A function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ (with κ a cardinal) *preserves integrability over finite measure spaces* if for every measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$ and all $(f_i \in \mathcal{L}^1(\mu))_{i\in\kappa}$ we have $\tau((f_i)_{i\in\kappa}) \in \mathcal{L}^1(\mu)$.

Theorem (A., 2020)

For every cardinal κ, a function $\tau\colon \mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability over finite measure spaces if and only if it is measurable and subaffine.

Subaffine := there are positive real numbers $k, \lambda_1, \ldots, \lambda_n$ such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$|\tau(\mathbf{x})| \leq k + \sum_{i=1}^{n} \lambda_i |x_i|.$$

i.e., τ...

- is at most linear in a neighbourhood of ∞;
- ~~is at most linear in a neighbourhood of 0;~~
- is bounded on bounded sets.

The function $\sqrt{|-|}\colon \mathbb{R} \to \mathbb{R}$ is measurable, (not sublinear, but) subaffine. Hence, it preserves integrability over finite measure spaces.

The constant function $1\colon \mathbb{R}^0 \to \mathbb{R}$ preserves integrability over finite measure spaces.

Question 2:
set of generators?

Theorem (A., 2020)

The infinitary clone on $\mathbb{R}$ of functions that preserve integrability (= measurable + sublinear) is generated by

- *"linear" operations:*
  - 0;
  - $+$;
  - *for $\lambda \in \mathbb{R}$, the scalar multiplication by λ, i.e. $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda \cdot x$;*
- *the truncation by 1, i.e. $\mathbb{R} \to \mathbb{R}$, $x \longmapsto x \wedge 1$;*
- *the truncated countable supremum*

$$(y, x_1, x_2, \dots) \mapsto \sup_{n=1}^{\infty}\{\min\{y, x_n\}\}.$$

Theorem (A., 2020)

The infinitary clone on $\mathbb{R}$ of functions that preserve integrability over finite measure spaces (= measurable + subaffine) is generated by

- *"affine" operations:*
  - 0;
  - $+$;
  - *for $\lambda \in \mathbb{R}$, the scalar multiplication by λ, i.e. $\mathbb{R} \to \mathbb{R}$, $x \mapsto \lambda \cdot x$;*
  - 1;
- *the truncated countable supremum*

$$(y, x_1, x_2, \dots) \mapsto \sup_{n=1}^{\infty}\{\min\{y, x_n\}\}.$$

Question 3:
axiomatization?

Recall: a function $\mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability over *finite* measure spaces if and only if it is measurable and subaffine.

Theorem (A., 2020)

The variety generated by the clone of measurable subaffine functions on $\mathbb{R}$ is the class of Dedekind σ-complete vector lattice with a weak unit.

Vector lattice (a.k.a. *Riesz space*) := vector space + lattice order + compatibility conditions.

Dedekind σ-complete := every countable subset with an upper bound has a supremum. (Equivalently, for vector lattices: every countable subset with a lower bound has an infimum.)

Weak unit := element $1 \geq 0$ such that $x \wedge 1 = 0$ implies $x = 0$.

The structure of a Dedekind σ-complete vector lattice with a weak unit is the richest algebraic structure shared by all $\mathcal{L}^1(\mu)$ with μ finite (and that uses only pointwise applications of functions $\mathbb{R}^\kappa \to \mathbb{R}$).

Recall: a function $\mathbb{R}^\kappa \to \mathbb{R}$ preserves integrability if and only if it is measurable and sublinear.

Theorem (A., 2020)

The variety generated by the clone of measurable sublinear functions on $\mathbb{R}$ *is the class of Dedekind* σ*-complete truncated vector lattices.*

Truncated [Ball, 2014] $:=$ ... $=$ with a unary endofunction behaving like a meet with a weak unit $a \mapsto a \wedge 1$.

The structure of a Dedekind σ-complete truncated vector lattice is the richest algebraic structure shared by all $\mathcal{L}^1(\mu)$'s (and that uses only pointwise applications of functions $\mathbb{R}^\kappa \to \mathbb{R}$).

Theorem (A., 2020)

Dedekind σ-complete vector lattice with a weak unit form a variety of algebras, generated by $\mathbb{R}$.

$$\{\text{Ded. } \sigma\text{-compl. vect. latt. w. weak unit}\} = \mathbb{HSP}(\mathbb{R})$$

Theorem (A., 2020)

Dedekind σ-complete vector lattice with a weak unit form a variety of algebras, generated by $\mathbb{R}$.

$$\{\text{Ded. } \sigma\text{-compl. vect. latt. w. weak unit}\} = \mathbb{HSP}(\mathbb{R}) = \mathbb{ISP}_{\sigma\text{-reduced}}(\mathbb{R}).$$

In fact, every Dedekind σ-complete vector lattice with a weak unit is a subalgebra of a power of $\mathbb{R}$ modulo a σ-ideal.

($\approx$ Loomis-Sikorski theorem.)

Analogous results hold for Dedeking σ-complete truncated vector lattices.

Theorem (A., 2020)

The free Dedekind σ-complete vector lattice with a weak unit over a set I (exists, and) is

$$\{\textit{measurable and subaffine } \mathbb{R}^I \to \mathbb{R}\}.$$

Theorem (A., 2020)

The free Dedekind σ-complete truncated vector lattice over a set I (exists, and) is

$$\{\textit{measurable and sublinear } \mathbb{R}^I \to \mathbb{R}\}.$$

If we look at the functions $[0,1]^I \to [0,1]$ that preserve integrability over finite measure spaces (or, equivalently, over probability spaces), we can discard the subaffine condition and we are left with the measurable one.

Theorem (Di Nola, Lapenta, Lenzi, 2021)

The free Dedekind σ-complete Riesz MV-algebra over a set I is

$$\{\textit{Baire measurable } [0,1]^I \to [0,1]\}.$$

Baire measurable $:=$ measurable with respect to the σ-algebra of Baire sets of $[0,1]^I$, i.e., the smallest σ-algebra containing all zerosets of continuous functions from $[0,1]^I$ with the product topology to $[0,1]$. It coincides with the product σ-algebra of $[0,1]^I$.

To sum up

$\mathcal{L}^1(\mu)$'s: Dedekind σ-complete truncated vector lattice.

Finite $\mathcal{L}^1(\mu)$'s: Dedekind σ-complete vector lattice with a weak unit.

Free Dedekind σ-complete truncated vector lattice over I:

$$\left\{\text{measurable and sublinear } \mathbb{R}^I \to \mathbb{R}\right\}.$$

Free Dedekind σ-complete vector lattice with a weak unit over I:

$$\left\{\text{measurable and subaffine } \mathbb{R}^I \to \mathbb{R}\right\}.$$

$\mathcal{L}^1(\mu)$'s: Dedekind σ-complete truncated vector lattice.

Finite $\mathcal{L}^1(\mu)$'s: Dedekind σ-complete vector lattice with a weak unit.

Free Dedekind σ-complete truncated vector lattice over I:

$$\{\text{measurable and sublinear } \mathbb{R}^I \to \mathbb{R}\}.$$

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Thank you!

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