

# On the category of metric compact Hausdorff spaces

See recording at <https://youtu.be/7thZXynhztU>

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Based on:



Marco Abbadini, Dirk Hofmann.

Barr-coexactness for metric compact Hausdorff spaces.

*Theory and Applications of Categories* (2025). [Link](#).

# A category

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Metric is similar to Order (quantale-enrichment).

(Quantales are special symmetric strict monoidal categories.)

## Historical Background

Compact Hausdorff spaces = Eilenberg-Moore algebras for the ultrafilter monad on **Set**.

**CompHaus** is a variety of infinitary algebras; it has an algebraic flavor. For example, it is Barr-exact.

**CompHaus** has also a topological flavor.

Usually, the opposite of a category with a topological flavor has an algebraic flavor:

- ▶ While **Top** is not regular, **Top**<sup>op</sup> is regular (in fact, a quasivariety of infinitary algebras [Barr, Pedicchio, 1995]).
- ▶ The opposite of the category of Stone spaces is a variety of algebras (Boolean algebras) [Stone, 1936]:

$$\text{hom}_{\mathbf{Stone}}(-, \{0, 1\}): \mathbf{Stone}^{\text{op}} \rightarrow \mathbf{Set}$$

is monadic.

(We recall that every monadic functor to **Set** is representable, by the free algebra on one generator.)

Theorem ([Duskin, 1969], all details in [Barr, Wells, 1985])

*The functor*

$$\text{hom}_{\mathbf{CompHaus}}(-, [0, 1]): \mathbf{CompHaus}^{\text{op}} \rightarrow \mathbf{Set}$$

*is monadic.*

$\mathbf{CompHaus}^{\text{op}}$  is a variety of infinitary algebras: it has an algebraic flavor.

$\mathbf{CompHaus}$  is complete, cocomplete, and Barr-coexact,  $[0, 1]$  is regular injective and a regular cogenerator (= Urysohn's lemma).

Stone duality (between Stone spaces and Boolean algebras) has an important generalization to ordered-topological spaces: Priestley duality.

*Priestley space* := Stone space + compatible partial order.

Priestley duality [Priestley, 1970]: Priestley spaces are dual to bounded distributive lattices (which form a variety).

$$\text{hom}_{\mathbf{Priestley}}(-, \{0, 1\}) : \mathbf{Priestley}^{\text{op}} \rightarrow \mathbf{Set}$$

is monadic.

<p>Compact Hausdorff</p> <p><b>CompHaus</b><sup>op</sup> <math>\xrightarrow{\text{hom}(-, [0,1])}</math> <b>Set</b></p>	<p>Stone</p> <p><b>Stone</b><sup>op</sup> <math>\xrightarrow{\text{hom}(-, \{0,1\})}</math> <b>Set</b></p>
<p>Compact Hausdorff + order</p> <p>?</p>	<p>Stone + order</p> <p><b>Priestley</b><sup>op</sup> <math>\xrightarrow{\text{hom}(-, \{0,1\})}</math> <b>Set</b></p>

Question [Hofmann, Neves, Nora, 2018]: is there an analogue with “compact Hausdorff spaces + order” instead of “Stone spaces + order”?

*Nachbin space* (a.k.a. compact ordered space) [Nachbin, 1948]: compact Hausdorff space + compatible partial order.

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Theorem (A., 2019)

$$\text{hom}_{\text{Nachbin}}(-, [0, 1]): \mathbf{Nachbin}^{\text{op}} \rightarrow \mathbf{Set}$$

*is monadic.*

(See [A., Reggio, 2019] for a nicer proof.)

It is also a way to collect categorical properties of **Nachbin**:

1. **Nachbin** is complete and cocomplete [Tholen, 2009].
2.  $[0, 1]$  is a regular injective regular cogenerator [Nachbin, 1960].
3. **Nachbin** is Barr-coexact [A., Reggio, 2020]. (Coregularity and something more already in [Hofmann, Neves, Nora, 2018]).

## Separated metric compact Hausdorff spaces

From Lawvere, we know that order is similar to metric.

Is there an analogue in the metric setting?

Before recalling **separated metric compact Hausdorff spaces**, let us see some drawbacks of the category of classical **compact metric spaces** and non-expansive maps:

- ▶ **not cocomplete.**

Remedy: allow distance  $\infty$ .

- ▶ **not complete.**

Remedy: topology **compatible** with the metric, rather than **induced** by it.

## Definition

A *metric* on a set  $X$  is a map  $d: X \times X \rightarrow [0, \infty]$  satisfying:

- ▶ (reflexivity)  $d(x, x) = 0$ ;
- ▶ (triangle inequality)  $d(x, z) \leq d(x, y) + d(y, z)$ .

A metric is *separated* if  $d(x, y) = 0 = d(y, x)$  implies  $x = y$ .

All results are true also when restricted to the symmetric case ( $d(x, y) = d(y, x)$ .)

If  $d$  is only allowed to take values 0 and  $\infty$ ,

- ▶ metric = preorder (where  $d(x, y) = 0$  means  $x \leq y$ ),
- ▶ separated metric = partial order.

## Definition ([Hofmann, Reis, 2018])

*(Separated) metric compact Hausdorff space* := compact Hausdorff space  $X$  equipped with a lower semicontinuous (separated) metric  $X \times X \rightarrow [0, \infty]$ .

Lower semicontinuous:

$$d(x_0, y_0) \leq \liminf_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} d(x, y).$$

I.e.: small topological perturbations may yield great increments in distances, but not great decrements.

Equivalently, continuous wrt the topology generated by the sets  $(a, \infty]$ .

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**Example:** any compact metric space (in the classical sense).

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**Example:**  $[0, \infty]$ , with  $d(a, b) = \begin{cases} b - a & \text{if } a < b; \\ 0 & \text{otherwise.} \end{cases}$

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**Example:** For any set  $X$ ,  $[0, 1]^X$  with the product metric (= sup metric) and the product topology.

Metric compact Hausdorff spaces are the algebras for the “metric ultrafilter” monad on the category of metric spaces and nonexpansive maps (See [Hofmann, Reis, 2018], building on [Tholen, 2009]).

**MetCH<sub>sep</sub>** := category of separated metric compact Hausdorff spaces  
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## Question

Is

$$\text{hom}_{\text{MetCH}_{\text{sep}}}(-, [0, \infty]): \text{MetCH}_{\text{sep}}^{\text{op}} \rightarrow \text{Set}$$

monadic?

I.e.:

1. Is  $\mathbf{MetCH}_{\text{sep}}$  complete and cocomplete?
2. Is  $\mathbf{MetCH}_{\text{sep}}$  Barr-coexact?
3. Is  $[0, \infty]$  a regular injective regular cogenerator of  $\mathbf{MetCH}_{\text{sep}}$ ?

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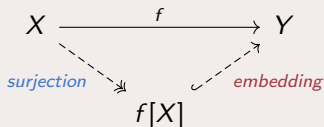
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? (*Future work.*)

## Theorem ([A., Hofmann, 2025])

1.  $\mathbf{MetCH}_{\text{sep}}$  has (*epi*, *regular mono*) factorization:



equip  $f[X]$  with the metric and topology induced by  $Y$

2. *epis* = *surjective morphisms*;
3. *regular monos* = *strong monos* = *embeddings*.

## Theorem ([A., Hofmann, 2025])

$\mathbf{MetCH}_{\text{sep}}$  is coregular.

$$\begin{array}{ccc} K & \hookrightarrow & X \\ f \downarrow & & \downarrow \\ K' & \dashrightarrow & X' \end{array} \quad \sqsupset$$

Given an embedding  $K \hookrightarrow X$  and a morphism  $f: K \rightarrow K'$ , inside  $X$  we can replace  $K$  by a copy of  $K'$  (and appropriate adjustments outside of  $K'$  induced by  $f$ ).

## Theorem ([A., Hofmann, 2025])

**MetCH<sub>sep</sub>** is *Barr-coexact*.

For a separated metric compact Hausdorff space  $X$ , a closed subset  $K \subseteq X$  induces a quotient of  $X + X$  by gluing the two copies of  $K$ .

Barr-coexactness: every surjective morphism  $X + X \twoheadrightarrow Z$  satisfying coreflexivity, cosymmetry, cotransitivity (first-order conditions) arises in this way.

To sum up

**MetCH<sub>sep</sub>** := category of separated metric compact Hausdorff spaces and continuous non-expansive maps. [Hofmann, Reis, 2019]

Is

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monadic? I.e.:

1. Is **MetCH<sub>sep</sub>** complete and cocomplete? ✓ [Tholen, 2009]
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Thank you.