Natural Duality for Finitely Valued Algebras

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The beauty of dualities

- Dualities offer a second viewpoint on algebraic structures:

  geometric / spatial, rather than symbolic.

- Classical examples:
  - Stone: Boolean algebras $\leftrightsquigarrow$ Stone spaces
  - Priestley: BDL $\leftrightsquigarrow$ Priestley spaces

- An algebra corresponds to a structured space.

The power of dualities (beyond representation)

- Algebras in $\mathbb{ISP}(\mathbf{L})$ (the usual setting for such dualities) can already be **represented** as L-valued functions (with pointwise operations).
- A **duality** enriches this picture:
  - it **characterizes the canonical representations**,
  - it also represents **morphisms** in a natural way
    $$f\colon \mathbf{A} \to \mathbf{B} \quad \leftrightsquigarrow \quad \text{continuous structure-preserving map},$$
  - giving a **bijective correspondence**
    (both on objects and morphisms).
- This leads to a two-way **dictionary**:
  - products of algebras $\leftrightsquigarrow$ sums of spaces ("logarithmic" compression)
  - coproducts of algebras $\leftrightsquigarrow$ products of spaces (easier to describe)
  - congruences $\leftrightsquigarrow$ subspaces
  - free algebras $\leftrightsquigarrow$ powers of the dual of $\mathrm{Free}(1)$
  - algebraic questions $\leftrightsquigarrow$ geometric/topological ones

Our starting point: positive MV-algebras

- We were interested in obtaining a duality for **positive MV-algebras**:
  - the $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras,
  - i.e. the quasivariety generated by $[0, 1]$ with these operations.
- Here, the natural dualizing algebra would be $[0, 1]$.
- Dualities of a similar flavour exist for MV-algebras:
  - Cignoli–Dubuc–Mundici (2003): locally finite MV-algebras. Dualizing algebra: $[0, 1] \cap \mathbb{Q}$,
  - Cignoli–Marra (2012): weakly locally finite MV-algebras. Dualizing algebra: $[0, 1]$.

- $[0,1]$ is infinite.
- The general theory of natural dualities is well developed for **finite** dualizing algebras $\mathbf{L}$.
- There are extensions to some infinite $\mathbf{L}$, but they typically rely on equipping $\mathbf{L}$ with a **compact Hausdorff topology** and using it in the duality.
- The MV-dualities mentioned above follow a *different* pattern:
  - no topology is used on $\mathbf{L}$ (even for $\mathbf{L} = [0,1]$),
  - but at a price: the duality applies not to all of $\mathbb{ISP}(\mathbf{L})$, but to a **restricted class** of algebras.
- **Goal:** provide a natural-duality framework that generalizes this MV-style phenomenon to other infinite untopologized dualizing algebras $\mathbf{L}$.

A clue from the MV side

- In both Cignoli–Dubuc–Mundici (2003) and Cignoli–Marra (2012), the duality works because elements of the algebras behave like **functions with finite range** in $\mathbf{L}$.
- Intuition: think of $a \in \mathbf{A}$ as an $\mathbf{L}$-valued function on a Stone space, that takes only finitely many values in $\mathbf{L}$.
- This suggests that the right setting for infinite $\mathbf{L}$ is **finitely valued $\mathbf{L}$-algebras**.

A restrictive but natural class: finitely valued algebras

> *Finitely valued $\mathbf{L}$-algebra: an algebra $\mathbf{A}$ s.t. there is a set X and an embedding $\mathbf{A} \hookrightarrow \mathbf{L}^X$ s.t. each $f \in \mathbf{A}$ has* **finite image in L**.

I.e., it has *some* representation with functions of finite range. ($\mathbf{L}$-*algebra*: an alg. in $\mathbb{ISP}(\mathbf{L})$.)

Under mild conditions (present in our duality result), this is equivalent to:

> *Canonically finitely valued $\mathbf{L}$-algebra: an algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{L})$ s.t., for each $a \in \mathbf{A}$, the set $\{ h(a) \mid h \in \hom(\mathbf{A}, \mathbf{L}) \}$ is finite.*

I.e., *the canonical representation* $\mathbf{A} \hookrightarrow \mathbf{L}^{\hom(\mathbf{A},\mathbf{L})}$ is with functions of finite range.

Sanity checks:

- all finite powers $\mathbf{L}^n$ and their subalgebras are finitely $\mathbf{L}$-valued;
- when $\mathbf{L}$ is finite, "finitely valued $\mathbf{L}$-algebra" = "in $\mathbb{ISP}(\mathbf{L})$".

Viewpoint via class operators

- For a dualizing algebra $\mathbf{L}$, standard natural dualities work with the whole $\mathbb{ISP}(\mathbf{L})$ = algebras built from $\mathbf{L}$ by isomorphisms, subalgebras and arbitrary products.
- For an arbitrary (possibly infinite, non-topologized) $\mathbf{L}$, we replace $\mathbb{P}(\mathbf{L})$ (all powers $\mathbf{L}^X$) by the class of **finite-range powers**:

$$\mathrm{FinRng}(X, \mathbf{L}) \coloneqq \{\, f \colon X \to \mathbf{L} \mid f[X] \text{ is finite} \,\} \leq \mathbf{L}^X.$$

  Define:

$$\mathbb{P}^{\mathrm{fr}}(\mathbf{L}) \coloneqq \{\, \mathrm{FinRng}(X, \mathbf{L}) \mid X \text{ any set} \,\}.$$

- Our duality will apply to $\mathbb{ISP}^{\mathrm{fr}}(\mathbf{L})$ (= the class of finitely valued $\mathbf{L}$-algebras), not to $\mathbb{ISP}(\mathbf{L})$. (For $\mathbf{L}$ finite, they coincide.)

Assumptions on the dualizing algebra $\mathbf{L}$

Think of $\{0,1\}$ (Bool. alg. / bdd. distr. lattice) or $[0,1]$ (MV-algebra).

- $\mathbf{L}$ has at least **two distinct constant symbols**.
- $\mathbf{L}$ has only **trivial partial endomorphisms**.

  I.e., for any $\mathbf{A} \leq \mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism $\mathbf{A} \to \mathbf{L}$.
- $\mathbf{L}$ has a $(k+1)$-ary **near-unanimity** term, with $k \geq 2$.

  (E.g.: $\mathbf{L}$ has a lattice reduct; $\to$ majority term, i.e. ternary near-unanimity term).

Remarks.

- Under these hypotheses, "finitely valued algebras" = "canonically finitely valued algebras". Moreover, the dual structure is purely relational + topological (no function symbols).
- The $(k+1)$-ary **near-unanimity** term allows the dual space to be a Stone space + k-ary constraints: e.g., in Priestley spaces, the order is a set of binary constraints.

Duality theorem (finitely valued setting)

Main result (A., Přenosil)

Let **L** *satisfy the assumptions on the previous slide. Then the category*

$$\mathbb{ISP}^{\mathrm{fr}}(\mathbf{L}) \quad (\textit{finitely valued } \mathbf{L}\textit{-algebras and homomorphisms})$$

is dually equivalent to the category of k-**ary L-Priestley spaces** *(and appropriate maps).*

In particular:

- if $k = 2$ (majority), the dual structure is driven by **binary** local constraints (Priestley-flavoured);
- if **L** is finite, $\mathbb{P}^{\mathrm{fr}}(\mathbf{L}) = \mathbb{P}(\mathbf{L})$ and we recover the usual natural-duality scope.

k-ary $\mathbf{L}$-Priestley spaces (definition)

A k-**ary** $\mathbf{L}$**-Priestley space** consists of:

- a Stone space X;
- for every $I \subseteq X$ with $|I| \leq k$, a **subalgebra** $\mathbf{A}_I \leq \mathbf{L}^I$, thought of as the set of *admissible* $\mathbf{L}$*-valued local functions* (on I).

These data satisfy:

- **Separation:** for $x \neq y \in X$ there is $f \in \mathbf{A}_{\{x,y\}}$ s.t. $f(x) \neq f(y)$.
- **Extension:** for $I \subseteq X$ with $|I| \leq k$ and $f \in \mathbf{A}_I$, there is a continuous $g \colon X \to \mathbf{L}$ (with $\mathbf{L}$ discrete) s.t., for all $J \subseteq X$ with $|J| \leq k$, $g|_J \in \mathbf{A}_J$.

Comment. When $k = 2$ (majority), the structure is determined by **binary** constraints $\mathbf{A}_{\{x,y\}}$, echoing the Priestley paradigm.

Example: recovering Priestley when $L = \mathbf{2}$, $k = 2$

Let $\mathbf{L} = \mathbf{2} = \{0, 1\}$ (with $\wedge, \vee, 0, 1$), which has a majority term, so $k = 2$. For a Priestley space $(X, \leq)$ define, for each $I \subseteq X$ with $|I| \leq 2$,

$$\mathbf{A}_I := \{\, f : I \to \mathbf{2} \mid f \text{ is order-preserving} \,\} \;\leq\; \mathbf{2}^I.$$

Then $(X, (\mathbf{A}_I)_I)$ is a 2-ary **2**-Priestley space:

- **Separation:** if $x \neq y$, then either $x \nleq y$ or $y \nleq x$. In the former case, take $x \mapsto 1$ and $y \mapsto 0$, otherwise take $x \mapsto 0$ and $y \mapsto 1$.
- **Extension:** if $f \in \mathbf{A}_I$ with $|I| \leq 2$, then f extends to a continuous $g \colon X \to \mathbf{2}$ with $g|_J \in \mathbf{A}_J$ for all $|J| \leq 2$ (this is Priestley's separation axiom).

From k-ary $\mathbf{L}$-Priestley spaces to finitely valued algebras

The functor maps a k-ary $\mathbf{L}$-Priestley space $(X, (\mathbf{A}_I)_{I \in [X]^{\leq k}})$ to

$$\left\{ f \colon X \to \mathbf{L} \;\middle|\; f \text{ cont. (with } \mathbf{L} \text{ discrete)}, \ f|_I \in \mathbf{A}_I \text{ for all } I \in [X]^{\leq k} \right\}.$$

It is a finitely valued $\mathbf{L}$-algebra: X compact $+$ $\mathbf{L}$ discrete $\Rightarrow$ a continuous $X \to \mathbf{L}$ has finite image.

E.g., in the Priestley case, we get

$$\{ f \colon X \to \mathbf{2} \mid f \text{ continuous and order-preserving} \},$$

i.e. the lattice of clopen upsets of $(X, \leq)$.

For the MV-algebras $\mathbf{L} = [0,1] \cap \mathbb{Q}$ and $\mathbf{L} = [0,1]$, our duality gives the dualities of Cignoli–Dubuc–Mundici for locally finite MV-algebras, and of Cignoli–Marra for weakly locally finite MV-algebras.

(This is after some further simplifications that make possible turning binary constraints into unary constraints in certain cases, which we describe in our preprint.)

Main result (A., Přenosil)

Let $\mathbf{L}$ be an algebra such that:

- $\mathbf{L}$ has at least **two distinct constant symbols**;
- $\mathbf{L}$ has only **trivial partial endomorphisms**.

  I.e., for any $\mathbf{A} \leq \mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism $\mathbf{A} \to \mathbf{L}$.
- $\mathbf{L}$ has a $(k+1)$-ary **near-unanimity** term, with $k \geq 2$.

  (E.g.: $\mathbf{L}$ has a lattice reduct; $\to$ majority term, i.e. ternary near-unanimity term).

Then the category

$$\mathbb{ISP}^{\mathrm{fr}}(\mathbf{L}) \quad \text{(}\textbf{finitely valued L-algebras} \text{ and homomorphisms)}$$

is dual to the category of k-**ary L-Priestley spaces**.

Thank you!

ArXiv: *Abbadini, Přenosil, Duality for finitely valued algebras*