The doctrinal Herbrand's theorem and its Stone dual

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6^{th} ItaCa Workshop
Milan, Italy

22 December 2025

Joint work in progress with Francesca Guffanti

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1. Herbrand's theorem

Herbrand's theorem (1930) describes the validity of an existential formula $\exists x\, \phi(x)$ (with $\phi(x)$ quantifier-free) in terms of the validity of quantifier-free formulas:

$$\vdash \exists x\, \phi(x) \iff \text{there are } c_1, \dots, c_n \text{ s.t.} \quad \vdash \phi(c_1) \vee \cdots \vee \phi(c_n).$$

1. Herbrand's theorem

Herbrand's theorem (1930) describes the validity of an existential formula $\exists x\, \phi(x)$ (with $\phi(x)$ quantifier-free) in terms of the validity of quantifier-free formulas:

$$\mathcal{T} \vdash \exists x\, \phi(x) \iff \text{there are } c_1, \ldots, c_n \text{ s.t. } \mathcal{T} \vdash \phi(c_1) \vee \cdots \vee \phi(c_n).$$

It holds modulo any universal theory $\mathcal{T}$ (i.e.: made of universal closures $\forall \underline{x}\, \alpha(\underline{x})$ of q.f. formulas, such as the theory of preorders, the theory of partial orders, any quasivariety of algebras).

2. First-order Boolean doctrines

Classical **propositional** logic	Classical **first-order** logic
$\Downarrow$	$\Downarrow$
Boolean algebras	*First-order Boolean doctrines* (Lawvere, 1969)

3. Stone duality

Classical **propositional** logic	Classical **first-order** logic
$\Downarrow$	$\Downarrow$
Boolean algebras	First-order Boolean doctrines
$\cong^{\mathrm{op}}$	$\cong^{\mathrm{op}}$
Stone spaces	Polyadic spaces (Joyal, 1971)

It is a duality between syntax (algebra of formulas) and semantics (spaces of models).

Aims

I will quickly address the question:

- What is the doctrinal reading of Herbrand's theorem? (algebras of formulas)

to then turn to the fun part:

- What is its Stone dual? (spaces of models)

The doctrinal reading of Herbrand's theorem

A *first-order Boolean doctrine (over* $\mathsf{FinSet}^{\mathrm{op}}$*)* is a functor

$$\mathbf{P}\colon \mathsf{FinSet} \to \mathsf{BA}$$

such that "adding dummy variables has a right and left adjoint" ($\forall$ and $\exists$), which moreover commute with substitutions (Beck-Chevalley).

For the talk: one sort, no function symbols, no $=$.

For a first-order theory $\mathcal{T}$:

$\mathbf{P}(X) = \{$FO formulas with free variables in $X\}/\mathcal{T}$-interprovability.

P on morphisms: simultaneous substitutions.

A bit more on Herbrand's theorem

Herbrand's theorem: for a universal theory $\mathcal{T}$:

$\mathcal{T} \vdash \exists x\, \phi(x) \iff$ there are $c_1, \ldots, c_n$ s.t. $\mathcal{T} \vdash \phi(c_1) \vee \cdots \vee \phi(c_n)$.

It <u>describes the validity</u> modulo a <u>universal theory</u> $\mathcal{T}$ of any formula of <u>quantifier alternation depth ≤ 1</u>, i.e.

$$\bigwedge \bigvee \alpha(\underline{z}),$$

with $\alpha(\underline{z})$ of the form

$$\exists \underline{x}\, \phi(\underline{x}, \underline{z}), \qquad \forall \underline{y}\, \psi(\underline{y}, \underline{z}),$$

with ϕ and ψ quantifier free.

Herbrand's theorem in doctrinal form

- Start with a **Boolean doctrine** (i.e. just a functor)

$$\mathbf{P}\colon \mathrm{FinSet} \to \mathrm{BA},$$

  modeling the class of **quantifier-free formulas** modulo a universal theory $\mathcal{T}$.

- $\mathbf{P}$ has a **quantifier completion**

$$\mathbf{P} \hookrightarrow \mathbf{P}^{\forall\exists},$$

  the universal way of freely adding first-order quantifiers. $\mathbf{P}^{\forall\exists}$ models the class of **all first-order formulas** modulo $\mathcal{T}$.

Herbrand's theorem in doctrinal form

- "Formulas of **quantifier alternation depth** ≤ 1":

$$\mathbf{P} \hookrightarrow \mathbf{P}_1 \hookrightarrow \mathbf{P}^{\forall\exists}$$

$\mathbf{P}_1$ = freely adding to $\mathbf{P}$ a layer of quantifier alternation depth.

- **Herbrand's thm** = explicit description of $\mathbf{P}_1$ *in terms of* $\mathbf{P}$.

A., Guffanti. Freely adding one layer of quantifiers to a Boolean doctrine. On arxiv. (2024)

See also

Wrigley. Existential completions and Herbrand's theorem. On arxiv. (2025)

Doctrinal Herbrand's thm: first step in the description of "free" first-order Boolean doctrines via <u>iterative addition of nested quantifiers</u>.

My interest in the Stone dual of Herbrand's thm

My interest in the Stone dual of Herbrand's thm

1. **An open problem.**

   Pitts' 1992: is every Heyting algebra the poset of subterminals of some topos?

   Pataraia announced a positive answer using

   $$\underline{\textit{iterative addition of nested quantifiers}} + \underline{\textit{duality}},$$

   but passed away before having left enough details.

   We start to develop the technology in the setting of classical first-order logic.

My interest in the Stone dual of Herbrand's thm

2. **An invitation.**

   We've been inspired by Gehrke's talk at CT 20$\to$21 (Genoa):

   doctrinal perspective + complexity of formulas + duality.

   (Gehrke, Jakl, Reggio.)

My interest in the Stone dual of Herbrand's thm

3. **A success case.**

   Dualities simplify free constructions: Ghilardi used

   $$\underline{\textit{iterative addition of nested implications}} + \underline{\textit{duality}}$$

   to describe free finitely generated Heyting algebras.

The Stone dual of Herbrand's theorem

Stone duality and doctrines

Composing a first-order Boolean doctrine

$$\mathbf{P}\colon \mathsf{FinSet} \to \mathsf{BA}$$

with Stone duality

$$\mathsf{BA} \cong \mathsf{Stone}^{\mathrm{op}}$$

gives...

A *polyadic space (over* $\mathsf{FinSet}^{\mathrm{op}}$*)* is a functor

$$\mathbf{E} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$$

with the following properties.

- Openness ($\leftrightarrow$ existence of adjoints).
- Amalgamation ($\leftrightarrow$ Beck-Chevalley).

Going back to

Joyal. Polyadic spaces and elementary theories. (1971)

See also Marquès' PhD thesis and

van Gool, Marquès. On duality and model theory for polyadic spaces. (2024)

Given a first-order theory $\mathcal{T}$ in a relational language, we have

$$\begin{aligned} \mathsf{E}\colon \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \mathrm{Mod}_X(\mathcal{T})_{/\equiv_{\mathrm{FO}}} \end{aligned}$$

with

$$\mathrm{Mod}_X(\mathcal{T}) := \{(M, \nu) \mid M \text{ model of } \mathcal{T},\ \nu\colon X \to M \text{ map}\}$$

where $(M, \nu) \equiv_{\mathrm{FO}} (M', \nu')$ if they are *elementarily equivalent*, i.e. they satisfy the same first-order formulas with free variables in X.

This is a polyadic space (and, vice versa, they are all of this form).

$$\begin{array}{c}\mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \\ \wr\Vert \\ \mathbf{E}\end{array}$$

- Let $\mathbf{E}\colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$ be a functor.

$$\begin{array}{c} \mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \\ \cong \\ \mathbf{E} \\ \\ \\ \mathbf{P} \\ \cong \\ \mathrm{Form}_{\mathrm{q.f.}}(-)_{/\dashv\vdash_{\mathcal{T}}} \end{array}$$

- Let $\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$ be a functor.
- Let $\mathbf{P}\colon \mathrm{FinSet} \to \mathrm{BA}$ be its Stone dual.

$$
\begin{tikzcd}
\mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \arrow[d, phantom, "\cong" rotate=90] \\
\mathbf{E}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{P} \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathbf{P}_1 \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathbf{P}^{\forall\exists} \arrow[d, phantom, "\cong" rotate=90] \\
\mathrm{Form}_{\mathrm{q.f.}}(-)_{/\dashv\vdash_{\mathcal{T}}} \arrow[r, hook] & \mathrm{Form}_{\mathrm{QA}\leq 1}(-)_{/\dashv\vdash_{\mathcal{T}}} \arrow[r, hook] & \mathrm{Form}_{\mathrm{FO}}(-)_{/\dashv\vdash_{\mathcal{T}}}
\end{tikzcd}
$$

- Let $\mathbf{E}\colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$ be a functor.
- Let $\mathbf{P}\colon \mathsf{FinSet} \to \mathsf{BA}$ be its Stone dual.

Herbrand's thm $=$ description of $\mathbf{P}_1$ in terms of $\mathbf{P}$.

$$
\begin{tikzcd}
\mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\text{q.f.}}} \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\text{QA}\leq 1}} \arrow[l, two heads] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Mod}_{(-)}(\mathcal{T})_{/\equiv_{\text{FO}}} \arrow[l, two heads] \arrow[d, phantom, "\cong" rotate=-90] \\
\mathbf{E} & \mathbf{E}_1 \arrow[l, two heads] & \mathbf{E}^{\forall\exists} \arrow[l, two heads] \\
\mathbf{P} \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=-90] & \mathbf{P}_1 \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=-90] & \mathbf{P}^{\forall\exists} \arrow[d, phantom, "\cong" rotate=-90] \\
\mathrm{Form}_{\text{q.f.}}(-)_{/\dashv\vdash_{\mathcal{T}}} \arrow[r, hook] & \mathrm{Form}_{\text{QA}\leq 1}(-)_{/\dashv\vdash_{\mathcal{T}}} \arrow[r, hook] & \mathrm{Form}_{\text{FO}}(-)_{/\dashv\vdash_{\mathcal{T}}}
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- Let $\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$ be a functor.
- Let $\mathbf{P}\colon \mathrm{FinSet} \to \mathrm{BA}$ be its Stone dual.

Herbrand's thm $=$ description of $\mathbf{P}_1$ in terms of $\mathbf{P}$.

Stone dual of Herbrand's thm $=$ descript. of $\mathbf{E}_1$ in terms of $\mathbf{E}$.

$$\mathbf{E}_1(\varnothing) \cong \mathrm{Mod}(\mathcal{T})_{/\equiv_{\mathrm{QA}\leq 1}}.$$

Theorem (The Stone dual of Herbrand's theorem)

$\mathbf{E}_1(\varnothing)$ *is the Stone space of* *Herbrand types for* $\mathbf{E}$.

Definition

A *Herbrand type for a functor* $\mathbf{E}\colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$ is a subfunctor of $\mathbf{E}$ mapping finite products (of $\mathsf{FinSet}^{\mathrm{op}}$) to quasi-products.

Quasi-product $:=$ the morphism to the product is epi.

Herbrand type: tuple $(F_X)_{X \in \mathsf{FinSet}}$, with $F_X \subseteq \mathbf{E}(X)$ closed, s.t.

1. $(F_X)_X$ is closed under substitution;
2. $F_{X_1 \sqcup X_2}$ is a quasi-product of F_{X_1} and F_{X_2};
3. $F_\varnothing$ is quasi-terminal (i.e., nonempty).

$$\mathbf{E}_1(\varnothing) \cong \mathrm{Mod}(\mathcal{T})_{/\equiv_{\mathrm{QA}\leq 1}}.$$

Theorem (The Stone dual of Herbrand's theorem)

$\mathbf{E}_1(\varnothing)$ *is the Stone space of* <u>*Herbrand types for* $\mathbf{E}$</u>*.*

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2. $F_{X_1 \sqcup X_2}$ is a quasi-product of F_{X_1} and F_{X_2};
3. $F_\varnothing$ is a singleton.

Theorem

Let $\mathcal{T}$ be a universal theory, and $\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$ *the functor mapping X to*

$$\mathbf{E}(X) = \mathrm{Mod}_X(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} := \{(M, X \to M) \mid M \in \mathrm{Mod}(\mathcal{T})\}_{/\equiv_{\mathrm{q.f.}}}.$$

Then,

$$\mathrm{Mod}(\mathcal{T})_{/\equiv_{\mathrm{QA}\leq 1}} \cong \{\textit{Herbrand types for } \mathbf{E}\}.$$

$$[M]_{\equiv_{\mathrm{QA}\leq 1}} \longmapsto \left(X \mapsto \overline{\left\{ \left[(M, X \to M) \right]_{\equiv_{\mathrm{q.f.}}} \right\}} \right).$$

We have been inspired by a similar idea in

Gehrke, Jakl, Reggio. A cook's tour of duality in logic: from quantifiers, through Vietoris, to measures. (2025)

To sum up

The Stone dual of Herbrand's theorem: Given a functor

$$\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone},$$

we describe

$$\mathbf{E}_1\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone},$$

(freely adding to **E** one layer of QA). In particular:

$$\mathbf{E}_1(\varnothing) = \{\text{Herbrand types for } \mathbf{E}\}$$

Herbrand type: subfunctor, finite products $\mapsto$ quasi-products.

- The Stone space $\mathbf{E}_1(X)$, for any X, is defined similarly.
- Not just $\mathrm{FinSet}^{\mathrm{op}}$, but any category with finite products (i.e., allowing function symbols and multiple sorts).
- With equality: same construction.

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- Not just $\mathsf{FinSet}^{\mathrm{op}}$, but any category with finite products (i.e., allowing function symbols and multiple sorts).
- With equality: same construction.

Thank you!

Appendix

Definition

Given a category C with finite products, a *first-order Boolean doctrine over* C is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with the following properties.

1. (Universal) For all $X, Y \in \mathsf{C}$,

$$\mathbf{P}(\mathrm{pr}_X^{X\times Y})\colon \mathbf{P}(X) \to \mathbf{P}(X \times Y)$$

   has a right adjoint $(\forall Y)_X$.

2. (Beck-Chevalley) For any $f\colon X' \to X$,

$$\begin{tikzcd} X & \mathbf{P}(X\times Y) \arrow[r, "(\forall Y)_X"] \arrow[d, "\mathbf{P}(f\times \mathrm{id}_Y)"'] & \mathbf{P}(X) \arrow[d, "\mathbf{P}(f)"] \\ X' \arrow[u, "f"] & \mathbf{P}(X'\times Y) \arrow[r, "(\forall Y)_{X'}"'] & \mathbf{P}(X'). \end{tikzcd}$$

A *polyadic space (over* $\mathrm{FinSet}^{\mathrm{op}}$*)* is a functor

$$\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$$

s.t.

- Openness ($\leftrightarrow$ adjoints): for all $X, Y \in \mathrm{FinSet}$, $\mathbf{E}(X \hookrightarrow X \sqcup Y)\colon \mathbf{E}(X \sqcup Y) \to \mathbf{E}(X)$ is an open map.
- Amalgamation ($\leftrightarrow$ Beck-Chevalley): For any $f\colon X \to X'$,

$$\begin{tikzcd} \mathbf{E}(X' \sqcup Y) \arrow[r, "\mathbf{E}(X' \hookrightarrow X' \sqcup Y)"] \arrow[d, "\mathbf{E}(f \sqcup \mathrm{id}_Y)"'] & \mathbf{E}(X') \arrow[d, "\mathbf{E}(f)"] \\ \mathbf{E}(X \sqcup Y) \arrow[r, "\mathbf{E}(X \hookrightarrow X \sqcup Y)"'] & \mathbf{E}(X) \end{tikzcd}$$

  is a quasi pullback (= pullback up to epi).

Theorem

Let $\mathcal{T}$ *be a universal theory, and* $\mathbf{E}\colon \mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$ *the functor mapping* X *to*

$$\mathbf{E}(X) = \mathrm{Mod}_X(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \coloneqq \{(M, X \to M) \mid M \in \mathrm{Mod}(\mathcal{T})\}_{/\equiv_{\mathrm{q.f.}}}.$$

Then,

$$\mathrm{Mod}(\mathcal{T})_{/\equiv_{\mathrm{QA}_{\leq 1}}} \cong \{\textit{Herbrand types for } \mathbf{E}\}.$$

$$[M]_{\equiv_{\mathrm{QA}_{\leq 1}}} \longmapsto \left(X \mapsto \overline{\left\{\left[(M, X \to M)\right]_{\equiv_{\mathrm{q.f.}}}\right\}}\right).$$

We found this idea in

Gehrke, Jakl, Reggio. A cook's tour of duality in logic: from quantifiers, through Vietoris, to measures. (2025)

1. $(F_X)_X$ is closed under substitution: For every $f\colon Y \to X$,

$$
\begin{tikzcd}
F_X \arrow[r, dashed] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & F_Y \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
\mathbf{E}(X) \arrow[r, "\mathbf{E}(f)"] & \mathbf{E}(Y)
\end{tikzcd}
$$

$\mathbf{E}(f)[F_X] \subseteq F_Y$.

Idea: An element in F_X is more or less of the form $(M, X \xrightarrow{\nu} M)$.

$$\mathbf{E}(f)(M, X \xrightarrow{\nu} M) = (M, Y \xrightarrow{\nu \circ f} M) \in F_Y.$$

2. $F_{X_1 \sqcup X_2}$ is the quasi-product of F_{X_1} and F_{X_2}:

$$
\begin{tikzcd}
 & & & y \arrow[dlll, dashed, mapsto] \arrow[drrr, dashed, mapsto] \arrow[d, phantom, sloped, "\ni"] & & & \\
x_1 \arrow[dr, phantom, sloped, "\ni"] & & & F_{X_1 \sqcup X_2} \arrow[dll] \arrow[drr] & & & x_2 \arrow[dl, phantom, sloped, "\in"] \\
 & F_{X_1} & & \mathbf{E}(X_1 \sqcup X_2) \arrow[u, phantom, sloped, "\subseteq"] \arrow[dl] \arrow[dr] & & F_{X_2} & \\
 & & \mathbf{E}(X_1) \arrow[ul, phantom, sloped, "\subseteq"] & & \mathbf{E}(X_2) \arrow[ur, phantom, sloped, "\supseteq"] & &
\end{tikzcd}
$$

For all X_1, X_2, $x_1 \in F_{X_1}$ and $x_2 \in F_{X_2}$ there is $y \in F_{X_1 \sqcup X_2}$ s.t. $\mathbf{E}(X_1 \hookrightarrow X_1 \sqcup X_2)(y) = x_1$ and $\mathbf{E}(X_2 \hookrightarrow X_1 \sqcup X_2)(y) = x_2$.

Idea: x_1 is more or less of the form $[(M, X_1 \xrightarrow{\nu_1} M)]$, x_2 is more or less of the form $[(M, X_2 \xrightarrow{\nu_2} M)]$. Then, one can take

$$y = \left[\left(M, \binom{X_1 \xrightarrow{\nu_1} M}{X_2 \xrightarrow{\nu_2} M} : X_1 \sqcup X_2 \to M\right)\right].$$

3. $F_\varnothing$ is a singleton.

Idea: it is $(M, \varnothing \to M)$.

The construction works also with equality in the language.

Empty language, with $=$, $\mathcal{T} := \{\forall x\, \forall y\, (x = y)\}$:

$$\begin{aligned} \mathbf{E} \colon \mathrm{FinSet}^{\mathrm{op}} &\longrightarrow \mathrm{Stone} \\ X &\longmapsto \mathrm{Mod}_X(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \cong \{*\} \end{aligned}$$

(because every q.f. formula is equivalent to $\top$ or $\bot$).

Two classes of models wrt $\equiv_{\mathrm{QA}\leq 1}$:

1. the class of singleton models (satisfying $\exists x \top$).
2. the class of the empty model (satisfying $\neg\exists x \top$).

In fact, **E** has two Herbrand types:

1. **E**;
2. $$\begin{aligned} \mathrm{FinSet}^{\mathrm{op}} &\longrightarrow \mathrm{Stone} \\ X &\longmapsto \begin{cases} \{*\} & \text{if } X = \varnothing; \\ \varnothing & \text{if } X \neq \varnothing. \end{cases} \end{aligned}$$

For the empty theory $\mathcal{T}$ in the empty language without "$=$",

$$\begin{aligned} \mathbf{E}\colon \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \mathrm{Mod}_X(\mathcal{T})_{/\equiv_{\mathrm{q.f.}}} \cong \{*\} \end{aligned}$$

(as the only quantifier-free formulas are $\top$ and $\bot$).

Two equivalence classes of models wrt $\equiv_{\mathrm{QA}\leq 1}$:

1. the class of nonempty models (satisfying $\exists x \top$).
2. the class of the empty model (satisfying $\neg\exists x \top$).

In fact, **E** has two Herbrand types:

1. **E**;
2. $$\begin{aligned} \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \begin{cases} \{*\} & \text{if } X = \varnothing; \\ \varnothing & \text{if } X \neq \varnothing. \end{cases} \end{aligned}$$