The doctrinal Herbrand's theorem and its Stone dual

See recording at `https://youtu.be/Y3LdW5PJoJ0`

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Herbrand's theorem

Herbrand's theorem (1930), in its weak form, describes the provability of an **existential** formula, such as $\exists x\, \varphi(x)$ with $\varphi(x)$ quantifier-free, in terms of the provability of **quantifier-free** formulas:

$$\mathcal{T} \vdash \exists x\, \varphi(x) \iff \text{there are ground terms } c_1, \dots, c_n \text{ such that}$$
$$\mathcal{T} \vdash \varphi(c_1) \vee \cdots \vee \varphi(c_n).$$

It holds relative to any universal theory $\mathcal{T}$, (i.e., axiomatized by universal closures of quantifier-free formulas).

Herbrand's theorem: provability of **existential** formulas in terms of provability of **quantifier-free** formulas.

Stone dual: description of the "space of models modulo satisfying the same **existential** formulas" in terms of the "space of models modulo satisfying the same **quantifier-free** formulas".

For the talk: one sort, only relation symbols, no equality.

Fix a universal theory $\mathcal{T}$.

For X a finite set, an X-*pointed model* of $\mathcal{T}$ is

$$(\underset{\text{model of } \mathcal{T}}{\underset{\uparrow}{M}}, \quad \underset{\text{function}}{\underset{\uparrow}{\nu}} \colon X \to M).$$

Here, we will look at two variations of the usual "elementary equivalence":

- $(M, \nu) \equiv_{\text{q.f.}} (M', \nu') \iff$ they satisfy the same quantifier-free formulas with free variables in X.
- $(M, \nu) \equiv_{\exists} (M', \nu') \iff$ they satisfy the same existential formulas with free variables in X.

Example: $\mathcal{T} := \varnothing$, in the empty language. A model is a set.

Let $X \in \mathrm{FinSet}$. Quantifier-free formulas with free variables in X: $\top, \bot$. All X-pointed models are equivalent with respect to $\equiv_{\mathrm{q.f.}}$.

Instead, the existential formula $\exists x \top$ separates the model $\varnothing$ from any nonempty model.

$$\mathrm{Mod}_X(\mathcal{T}) \coloneqq \{X\text{-pointed models of } \mathcal{T}\}.$$

$$\begin{aligned} \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \frac{\mathrm{Mod}_X(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} \end{aligned} \qquad \begin{aligned} \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \frac{\mathrm{Mod}_X(\mathcal{T})}{\equiv_{\exists}} \end{aligned}$$

On morphisms: $f\colon X \to Y$ is mapped to the function

$$[M,\ \nu\colon Y \to M] \longmapsto [M,\ \nu \circ f\colon X \to M].$$

Clopen: the set of pointed models satisfying a given q.f. formula / Boolean combination of existential formulas.

Stone dual of Herbrand's theorem: describe the second functor in terms of the first one.

Example: $\mathcal{T} := \varnothing$, in the empty language. A model is a set.

No q.f. formula separates pointed models. Thus,

$$\begin{aligned} \frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} \colon \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto 1. \end{aligned}$$

$\exists x \top$ separates the model $\varnothing$ from any nonempty model. Thus,

$$\begin{aligned} \frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\exists}} \colon \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ \varnothing &\longmapsto 2 \\ \text{nonempty } X &\longmapsto 1 \end{aligned}$$

We want to construct the functor $\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\exists}}$ from $\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}}$.

In this talk, I will focus just on the easiest bit of this construction: how to construct the Stone space

$$\frac{\mathrm{Mod}_{\varnothing}(\mathcal{T})}{\equiv_{\exists}}$$

in terms of the functor

$$\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}}.$$

Idea:

$$\frac{\mathrm{Mod}_{\varnothing}(\mathcal{T})}{\equiv_{\exists}} \hookrightarrow \left\{ \text{subfunctors of } \frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone} \right\}$$

$$\left[M\right]_{\equiv_{\exists}} \longmapsto \begin{array}{c} \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone} \\ X \mapsto \overline{\left\{ \left[M, \nu\right]_{\equiv_{\mathrm{q.f.}}} \mid \nu \colon X \to M \text{ map} \right\}} \end{array}$$

We characterize the image.

E.g.: with $\mathcal{T} = \varnothing$ in the empty language: Recall:

$$\begin{aligned} \frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\text{q.f.}}} \colon \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto 1. \end{aligned}$$

It has three subfunctors:

1. itself: $X \mapsto 1$;
2. the functor constantly $\varnothing$: $X \mapsto \varnothing$;
3. the functor $X \mapsto \begin{cases} 1 & \text{if } X = \varnothing, \\ \varnothing & \text{if } X \neq \varnothing. \end{cases}$

Idea:

$$\frac{\mathrm{Mod}_{\varnothing}(\mathcal{T})}{\equiv_{\exists}} \hookrightarrow \left\{\text{subfunctors of } \frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}\right\}$$

$$[\varnothing] \longmapsto X \mapsto \overline{\left\{\left[\varnothing, \nu\right]_{\equiv_{\mathrm{q.f.}}} \mid \nu \colon X \to \varnothing\right\}} = \begin{cases} 1 & \text{if } X = \varnothing \\ \varnothing & \text{if } X \neq \varnothing \end{cases}$$

$$[1] \longmapsto X \mapsto \overline{\left\{\left[1, \nu\right]_{\equiv_{\mathrm{q.f.}}} \mid \nu \colon X \to 1\right\}} = 1$$

There are three subfunctors, but only two of them come from the existential equivalence class of a model.

E.g.: the excluded subfunctor does not map $\varnothing$ to a singleton.

Theorem (The Stone dual of Herbrand's theorem)

For a universal theory $\mathcal{T}$ (in a relational 1-sorted language without equality),

$$\frac{\mathrm{Mod}_{\varnothing}(\mathcal{T})}{\equiv_{\exists}}$$

is homeomorphic to the Stone space of subfunctors of

$$\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$$

that map **finite coproducts** *of* FinSet *to* **quasi-products** *of* Stone.

Quasi-product := the morphism to the product is epi. (epi = reg epi = surjective in Stone.)

Left covering for finite products.

$(F_X)_{X \in \mathsf{FinSet}}$, with $F_X \subseteq \frac{\mathrm{Mod}_X(\mathcal{T})}{\equiv_{\mathrm{q.f.}}}$ closed, s.t.

1. $(F_X)_X$ is closed under substitution:
   if $[M, \nu \colon Y \to M] \in F_Y$ and $f \colon X \to Y$, then
   $[M, \nu \circ f \colon X \to M] \in F_X$;
2. $F_{X_1 \sqcup X_2} \longrightarrow\!\!\!\!\!\rightarrow F_{X_1} \times F_{X_2}$;
   Idea: an X-pointed model $(M, X \to M)$ and a Y-pointed model $(M, Y \to M)$ give an $(X \sqcup Y)$-pointed model $(M, X \sqcup Y \to M)$.
3. $F_\varnothing \longrightarrow\!\!\!\!\!\rightarrow 1$. $F_\varnothing$ is a singleton.
   Idea: every model M has an $\varnothing$-pointed model $(M, \varnothing \to M)$.

More generally, we describe

$$\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\exists}} : \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone},$$

in terms of

$$\frac{\mathrm{Mod}_{-}(\mathcal{T})}{\equiv_{\mathrm{q.f.}}} : \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}.$$

- With equality: same construction.
- Not just $\mathsf{FinSet}^{\mathrm{op}}$, but any category with finite products (i.e., allowing function symbols and multiple sorts).

We dualized the notion of "universal ultrafilter", used in our doctrinal version of Herbrand's theorem:

A., Guffanti. Freely adding one layer of quantifiers to a Boolean doctrine. Arxiv.

To sum up

Presheaf of Stone spaces: $\mathrm{FinSet}^{\mathrm{op}} \to \mathrm{Stone}$

Stone dual of Herbrand's theorem:

Subfunctors

Left coverings for finite products

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Save the date:

Saturday 26 – Sunday 27 September 2026