# Ideals

# and slices over the initial algebra

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# Motivating example I: commutative rings

$$\mathsf{CRing} \quad (+,\cdot,-,0,1) \quad \rightsquigarrow \quad \mathsf{CRng} \quad (+,\cdot,-,0).$$

$$\mathsf{CRing}/\mathbb{Z} \simeq \mathsf{CRng}.$$

The $\to$ direction:

$$\big(f\colon A \to \mathbb{Z}\big) \longmapsto f^{-1}[\{0\}].$$

The $\leftarrow$ direction: given a commutative rng $A$,

$$A \rtimes \mathbb{Z} \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}, \qquad (a,m) \sim a+m.$$

The fiber over 0 is $A$.

# Motivating example II: Boolean algebras

$$\mathsf{BA} \quad (\vee, \neg, 0) \quad \rightsquigarrow \quad \mathsf{GBA} \quad (\vee, \wedge, \smallsetminus, 0).$$

$$\mathsf{BA}/\{0,1\} \simeq \mathsf{GBA}.$$

The $\to$ direction:

$$\big(f\colon A \to \{0,1\}\big) \longmapsto f^{-1}[\{0\}].$$

The $\leftarrow$ direction: from $A \in \mathsf{GBA}$ one adjoins complements:

$$A \rtimes \{0,1\} \longrightarrow\!\!\!\!\!\rightarrow \{0,1\}, \qquad (a,0) \sim a, \qquad (a,1) \sim \neg a.$$

# The role of $\mathbb{Z}$ and $\{0, 1\}$

1. $\mathbb{Z}$ is the initial algebra of CRing, i.e., the algebra of nullary terms (up to identifying equivalent terms)
2. $\{0, 1\}$ is the initial algebra of BA, i.e., the algebra of nullary terms (up to identifying equivalent terms)

# Main question

Given

- a variety $\mathcal{V}$ (e.g., CRing, BA)
- and a nullary term $c$ (e.g., 0),

is there a canonical procedure to obtain a class of algebras (e.g., CRng, GBA) of "$c$-ideals", equivalent to the slice

$$\mathcal{V}/\mathbf{I}$$

over the initial algebra $\mathbf{I}$?

# The canonical signature attached to a constant

Fix a variety $\mathcal{V}$ (e.g., CRing or BA) and a nullary term $c$ (e.g., 0).

Take the set of *$c$-idempotent* terms:

$$\Sigma_c \coloneqq \{\vartheta \mid \mathcal{V} \vDash \vartheta(c, \dots, c) = c\}.$$

We regard $\Sigma_c$ as a signature.

For example, in Boolean algebras with $c = 0$,

$$x \vee y, \qquad x \wedge y, \qquad x \setminus y \coloneqq x \wedge \neg y, \qquad 0$$

are 0-idempotent, since

$$0 \vee 0 = 0, \qquad 0 \wedge 0 = 0, \qquad 0 \setminus 0 = 0, \qquad 0 = 0.$$

For any morphism $f\colon A \to \mathbf{I}$, the $c$-idempotent terms are precisely those under which the fiber $f^{-1}[\{c\}]$ is closed.

# The class $\mathcal{V}_c$

Define

$$\mathcal{V}_c := \text{the class of } \Sigma_c\text{-subreducts of algebras in } \mathcal{V}.$$

1. $\mathcal{V}_c$ is the class of preimages of $\{c\}$ under arbitrary $\mathcal{V}$-homomorphisms / under $\mathcal{V}$ homomorphisms to $\mathbf{I}$.
2. $\mathcal{V}_c$ is a variety, axiomatized by the $\Sigma_c$-equations holding in $\mathcal{V}$.

# The canonical functor

There is a functor

$$\begin{aligned}\mathcal{V}/\mathbf{I} &\longrightarrow \mathcal{V}_c \\ \bigl(f\colon A \to \mathbf{I}\bigr) &\longmapsto f^{-1}[\{c\}].\end{aligned}$$

Main question: when is it an equivalence of categories? I.e.: when is $f\colon A \to \mathbf{I}$ determined by the fiber over $c$?

# The equivalence criterion

## Main criterion

The functor

$$\mathcal{V}/\mathbf{I} \longrightarrow \mathcal{V}_c$$
$$\big(f \colon A \to \mathbf{I}\big) \longmapsto f^{-1}[\{c\}]$$

is an equivalence if and only if $\mathcal{V}$ is *classically c-ideal determined over constants*.

# What the condition says

$\mathcal{V}$ is *classically c-ideal determined over constants* $:=$ for every nullary $\mathcal{V}$-term $d$, there are finitely many unary $\mathcal{V}$-terms

$$\alpha_1(x), \dots, \alpha_n(x)$$

such that

$$\mathcal{V} \vDash \alpha_i(d) = c \qquad \text{for every } i \in \{1, \dots, n\},$$

and a $\mathcal{V}$-term $\vartheta(x_1, \dots, x_n)$ such that

$$\mathcal{V} \vDash \vartheta\big(\alpha_1(x), \dots, \alpha_n(x)\big) = x.$$

This condition is implied by the usual classical $c$-ideal determinacy.

# The unary case

Case $n = 1$: for a nullary term $d$, there are terms $\alpha(x)$ and $\vartheta(x)$ such that

$$\mathcal{V} \vDash \alpha(d) = c, \qquad \mathcal{V} \vDash \vartheta\alpha(x) = x.$$

Examples with $c = 0$:

$$\text{CRing}: \qquad \text{for } d \in \mathbb{Z}: \qquad \alpha(x) = x - d, \qquad \vartheta(x) = x + d.$$

BA :

| | $\alpha = \dots$ | $\vartheta = \dots$ |
|---|---|---|
| $d = 0$ | $\dots \mathrm{id}$ | $\dots \mathrm{id}$ |
| $d = 1$ | $\dots \neg$ | $\dots \neg$ |

# Recap up to now

$\mathcal{V}/\mathsf{I} \to \mathcal{V}_c$ is an equivalence iff $\mathcal{V}$ is classically $c$-ideal determined over constants.

Questions:

1. When can we get a tangible axiomatization of $\mathcal{V}_c$?
2. When is the quasi-inverse

$$\mathcal{V}_c \to \mathcal{V}/\mathsf{I}$$

given by a semidirect construction $- \rtimes \mathsf{I}$?

# Swappable constants

## Definition

A variety $\mathcal{V}$ has *swappable constants* if, for all nullary terms $d$ and $d'$ there are unary terms $\sigma_{d \mapsto d'}(x)$, $\sigma_{d' \mapsto d}(x)$ such that

$$\mathcal{V} \vDash \sigma_{d \mapsto d'}(d) = d', \qquad \mathcal{V} \vDash \sigma_{d' \mapsto d}(d') = d,$$

and

$$\mathcal{V} \vDash \sigma_{d \mapsto d'} \sigma_{d' \mapsto d}(x) = x = \sigma_{d' \mapsto d} \sigma_{d \mapsto d'}(x).$$

Examples:

$$\text{CRing}: \quad \sigma_{m \mapsto 0}(x) = x - m, \qquad \sigma_{0 \mapsto m}(x) = x + m.$$

Any variety with an Abelian group reduct has swappable constants.

$$\mathrm{BA}: \quad 0 \leftrightarrow 0 \text{ by } \mathrm{id}, \qquad 1 \leftrightarrow 0 \text{ by } \neg x.$$

# Consequences of swappability

Assume that $\mathcal{V}$ has swappable constants. Then,

1. For all $c$, $\mathcal{V}$ is classically $c$-ideal determined over constants; so,

$$\mathcal{V}/\mathbf{I} \simeq \mathcal{V}_c.$$

2. "All constants are the same": $\mathcal{V}_c$ and $\mathcal{V}_d$ are term-equivalent.
3. All fibers of a $\mathcal{V}$-morphism $f\colon A \to \mathbf{I}$ have the same cardinality;

   $$\text{as sets, } A \cong f^{-1}[\{c\}] \times \mathbf{I}.$$

   $\mathcal{V}_c \to \mathcal{V}/\mathbf{I}$ has a semidirect-product description $- \rtimes \mathbf{I}$.
4. An algorithm to compute a concrete presentation of $\mathcal{V}_c$. In particular, if $\mathcal{V}$ has a finite equational presentation and $\mathbf{I}$ is finite (as for BA) $\rightsquigarrow$ finite equational presentation of $\mathcal{V}_c$.

# To sum up

Given a variety $\mathcal{V}$ and a nullary term $c$.

$\mathcal{V}/\mathbf{I} \to \mathcal{V}_c$ is an equivalence iff
$\mathcal{V}$ is classically $c$-ideal determined over constants.

Special case: $\mathcal{V}$ has swappable constants. Then,

- an algorithm producing an explicit generating set for $c$-idempotent terms,
- an algorithm producing an explicit equational basis for $\mathcal{V}_c$,
- a semidirect-product construction for $\mathcal{V}_c \to \mathcal{V}/\mathbf{I}$.

*Thank you!*