On the symmetry between openness and compactness

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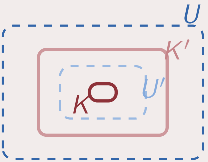
open sets $\longleftrightarrow$ *compact saturated sets*

"saturated" := "intersection of open sets".

In T_1 spaces (e.g., Hausdorff spaces), every subset is saturated.

Definition (Local compactness)



For compact K and open U with $K \subseteq U$, there are an open U' and a compact K' s.t.

$$K \subseteq U' \subseteq K' \subseteq U.$$

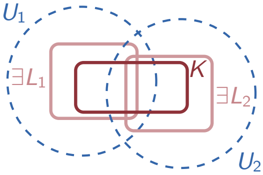
Swapping "open" $\leftrightarrow$ "compact" and $\subseteq \leftrightarrow \supseteq$.

Wilker's conditions

In every locally compact Hausdorff space:

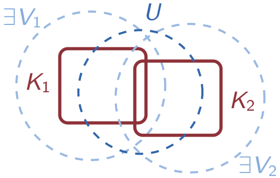
Open cover of a compact

$K \subseteq U_1 \cup U_2$
$\implies \exists L_1, L_2$ compact
$L_i \subseteq U_i, \quad K \subseteq L_1 \cup L_2.$



Compact intersection inside an open

$U \supseteq K_1 \cap K_2$
$\implies \exists V_1, V_2$ open
$V_i \supseteq K_i, \quad U \supseteq V_1 \cap V_2.$



Analogy:

$$\text{products} + \text{equalizers} \Longrightarrow \text{all limits}$$
$$\text{coproducts} + \text{coequalizers} \Longrightarrow \text{all colimits}$$

We get the second one via the opposite category.

Two theorems at the price of one.

If a class of spaces has an involution swapping opens and compact saturated sets, then $\rightsquigarrow$ two theorems at the price of one.

But the class of locally compact Hausdorff spaces is not symmetric. There are signs of symmetry, but not a full symmetry.

In locally compact Hausdorff spaces:

Open sets	*Compact sets*	
finite intersections are open	finite unions are compact	✓
binary unions are open	binary intersections are compact	✓
$\varnothing$ is open	X may fail to be compact	×

A subset K is compact iff, for every family $\mathcal{U}$ of open sets,

$$K \subseteq \bigcup_{U \in \mathcal{U}} U \quad \Longrightarrow \quad \exists U_1, \dots, U_n \in \mathcal{U}\ K \subseteq U_1 \cup \cdots \cup U_n.$$

Equivalently, it is enough to test directed covers:

For every directed family $\mathcal{I}$ of open sets,

$$K \subseteq \bigcup_{U \in \mathcal{I}} U \quad \Longrightarrow \quad \exists U \in \mathcal{I}\,(K \subseteq U).$$

Here “directed” means nonempty and any two elements have a common upper bound.

Every Hausdorff space satisfies the symmetric statement:

For every codirected family $\mathcal{F}$ of compact sets, for every open U,

$$\bigcap_{K \in \mathcal{F}} K \subseteq U \quad \Longrightarrow \quad \exists K \in \mathcal{F}\, (K \subseteq U),$$

The spaces satisfying this property (with compact *saturated* sets) are called well-filtered.

$$\text{Hausdorff} \Longrightarrow \text{sober} \Longrightarrow \text{well-filtered}.$$

Thus sober spaces already have a *partial* symmetry between openness and compactness.

Stone duality [Stone, 1936] (here, "duality" means "categorical duality"):

$$\text{Stone spaces} \longleftrightarrow \text{Boolean algebras}$$
$$\text{Spectral spaces} \longleftrightarrow \text{Bounded distributive lattices}$$

Bounded distributive lattice $\cong$ open compact sets.

The symmetry is perfect in both classes.

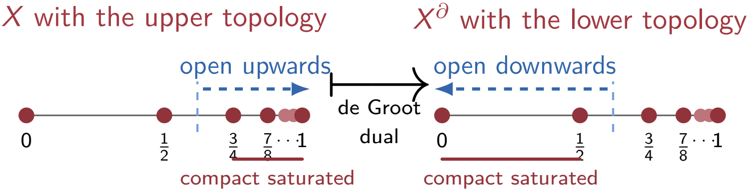
For a spectral space X, its *de Groot dual* X^∂ has the same underlying set and

$$\mathrm{Op}(X^\partial) = \{X \setminus K \mid K \in \mathrm{KSat}(X)\},$$

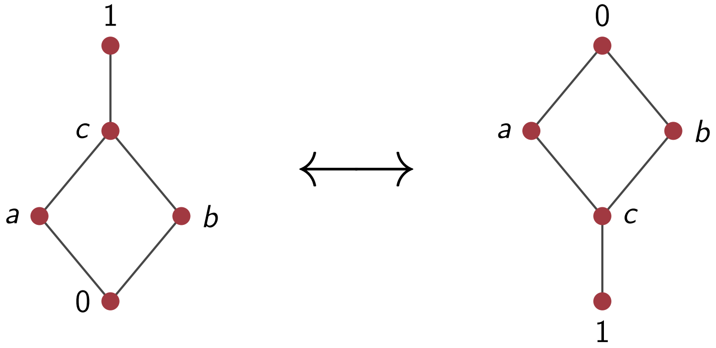
$$\mathrm{KSat}(X^\partial) = \{X \setminus U \mid U \in \mathrm{Op}(X)\}.$$

It is still a spectral space, and $(X^\partial)^\partial = X$.

$$X = \left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots, 1\right\} \subseteq [0,1].$$



Same underlying set; the open direction is reversed.


1
c
a
b
0
0
a
b
c
1

Another Stone-like duality is

$$\text{sober spaces} \quad \longleftrightarrow \quad \text{spatial frames.}$$

A sober space can be recovered from its poset of opens.

But sober spaces have only a partial symmetry

$$\text{open} \qquad \longleftrightarrow \qquad \text{compact saturated.}$$

Question

Can we enlarge the setting so that this duality sits inside a fully symmetric one?

We need a setting with a full symmetry between openness and compactness, and that is:

1. general enough to contain all sober spaces;
2. restrictive enough to keep a Stone-like duality.

Why I have been interested in this

- *Two-for-one*.
- *Representation of (not necessarily bounded) distributive lattices via compact opens*: in ordinary topological spaces, $\varnothing$ is always compact open.
- *Guidance*: preserving the symmetry helps identify the right definitions, proofs and constructions.
- *Aesthetic*.
- *Understanding*.

The price to pay

We take openness and compactness as *primitive* notions.

Reason: even for Hausdorff spaces, compact sets need not determine the open sets.

So, symmetrically, we should not ask open sets to determine compact sets either.

Two families

A space-like object will have

$\mathcal{K}$ "compact" sets and $\mathcal{O}$ "open" sets.

They interact, but neither is defined from the other.

Which closure properties survive?

To cover Hausdorff and sober spaces, we cannot require compact saturated sets to be closed under finite intersections.

Symmetrically, we do not require open sets to be closed under finite unions.

What we keep is:

$\mathcal{O}$	$\mathcal{K}$
closed under directed unions	closed under codirected intersections

Specialization order

For a topological space $(X, \mathcal{O})$, define

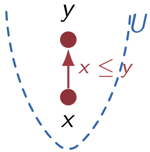
$$x \leq y \iff (\forall U \in \mathcal{O})\ x \in U \Rightarrow y \in U.$$

This is the *specialization order*; it is a partial order exactly when X is T_0.

With this order:

$$U \in \mathcal{O} \Rightarrow U \text{ is an upset},$$

$$A \text{ saturated} \iff A \text{ is an upset}.$$



$x \in U$ forces $y \in U$

Definition

A *ko-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}, \mathcal{O}$ sets of upsets of X s.t.

1. $\mathcal{O}$ is closed under directed unions;
   $\mathcal{K}$ is closed under codirected intersections.
2. $$K \subseteq \bigcup \mathcal{I} \Rightarrow \exists U \in \mathcal{I}\ (K \subseteq U)\ (K \in \mathcal{K},\ \mathcal{I} \subseteq \mathcal{O} \text{ directed}),$$
   $$\bigcap \mathcal{F} \subseteq U \Rightarrow \exists K \in \mathcal{F}\ (K \subseteq U)\ (U \in \mathcal{O},\ \mathcal{F} \subseteq \mathcal{K} \text{ codirected}).$$
3. For every $x \in X$,
   $$\uparrow x \in \mathcal{K}, \qquad X \setminus \downarrow x \in \mathcal{O}.$$

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Symmetry. If $(X, \mathcal{K}, \mathcal{O})$ is a ko-space, then so is

$$X^{\partial} = \left(X^{\mathrm{op}},\ \{X \setminus U \mid U \in \mathcal{O}\},\ \{X \setminus K \mid K \in \mathcal{K}\}\right).$$

$$(X^{\partial})^{\partial} = X.$$

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Hausdorff spaces. Let $(X, \mathcal{O})$ be Hausdorff. Then $(X, \mathcal{K}, \mathcal{O})$ is a ko-space by setting

$$\leq \; = \text{equality},$$
$$\mathcal{K} = \{\text{compact}\},$$
$$\mathcal{O} = \{\text{open}\}.$$

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Sober spaces. Let $(X, \mathcal{O})$ be sober. Then $(X, \mathcal{K}, \mathcal{O})$ is a ko-space by setting

$$\leq = \text{spec. ord.},$$
$$\mathcal{K} = \{\text{compact saturated}\},$$
$$\mathcal{O} = \{\text{open}\}.$$

Another example

On $\mathbb{R}$ with its usual order, one can take

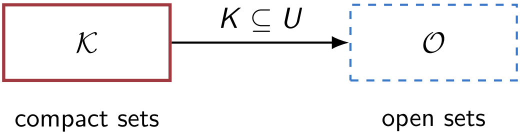
$$\mathcal{K} = \{\, [a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\varnothing\},$$

$$\mathcal{O} = \{\, (a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\mathbb{R}\}.$$

The pointfree side, informally

Given a ko-space $(X, \mathcal{K}, \mathcal{O})$, forget the points and remember:

$$(\mathcal{K}, \mathcal{O}, \subseteq).$$



This is analogous to replacing a sober space by its frame of opens, but now the open and compact sides are both remembered.

Main result

The category of ko-spaces (with appropriate morphisms) is dually equivalent to the category of *distributive bi-dcpos*.

$$\textit{ko-spaces} \quad \longleftrightarrow \quad \textit{distributive bi-dcpos}$$

$$(X, \mathcal{K}, \mathcal{O}) \quad \longleftrightarrow \quad (\mathcal{K}, \mathcal{O}, \lhd \subseteq \mathcal{K} \times \mathcal{O})$$

Distributive bi-dcpos are similar to *spatial frames*, but for the two-sorted setting:

one compact side $\quad + \quad$ one open side.

Sources of inspiration

- bitopology;
- formal concept analysis, polarities, Chu spaces, canonical extensions;
- domain theory and dcpos.

$$\begin{array}{ccc} \text{sober spaces} & \longleftrightarrow & \text{spatial frames} \\ \cap\!| & & \cap\!| \\ \text{ko-spaces} & \longleftrightarrow & \text{distributive bi-dcpos} \end{array}$$

Preprint:
Abbadini and Jung,
On the symmetry behind duality
arXiv:2507.18245

Thank you