Ideals and slices over the algebra of constants

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Varieties with good ideals

Classical sources:

Ursini (70's), Gumm–Ursini (80's), Aglianò–Ursini (90's).

Guiding examples

Unital rings	$\rightsquigarrow$	Rngs	(0-ideals)
Boolean algebras	$\rightsquigarrow$	Generalized Boolean algebras	(0-ideals)
MV-algebras	$\rightsquigarrow$	Wajsberg hoops	(1-ideals)

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Which operations should an ideal remember?

Fix a distinguished constant c.

A term $t(x_1, \ldots, x_n)$ is *c-idempotent* if

$$t(c, \ldots, c) \approx c.$$

These are the operations that can restrict to a c-ideal.

Example:

In unital rings:

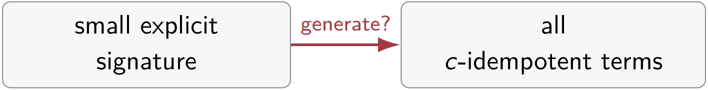
$$0 + 0 = 0.$$

In Boolean algebras:

$$0 \setminus 0 \approx 0, \qquad \text{but} \qquad \neg 0 \not\approx 0,$$

where $x \setminus y = x \wedge \neg y$.

When does a small signature suffice?



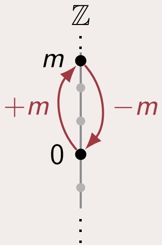
Sufficient criterion

We want to know when a few explicit c-idempotent operations generate every c-idempotent term.

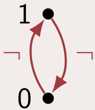
having swappable constants

Moving between constants

Unital rings



Boolean and MV-algebras



The algebra of constants

To state swappability uniformly, let $\mathbf{I}$ be the initial algebra of $\mathcal{V}$.

This is where the constants live: its elements are the term-definable constants, modulo equality in all algebras.

$\mathcal{V}$	$\mathbf{I}$
Unital rings	$\mathbb{Z}$
Boolean algebras	$\{0, 1\}$
MV-algebras	$\{0, 1\}$

Swappable constants

Definition

A variety $\mathcal{V}$ *has swappable constants* if, for all term-definable constants c and d, there are unary terms (called *shifts*)

$$\sigma_{c \mapsto d}(x), \qquad \sigma_{d \mapsto c}(x),$$

that are mutually inverse and such that

$$\sigma_{c \mapsto d}(c) \approx d, \qquad \sigma_{d \mapsto c}(d) \approx c.$$

Unital rings. For $d \in \mathbb{Z}$,

$$\sigma_{0 \mapsto d}(x) = x + d, \qquad \sigma_{d \mapsto 0}(x) = x - d.$$

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Boolean algebras.

$$\sigma_{0\mapsto 1}(x) = \neg x, \qquad \sigma_{1\mapsto 0}(x) = \neg x.$$

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Abelian group reduct. If $\mathcal{V}$ has an Abelian group reduct, then $\mathcal{V}$ has swappable constants:

$$\sigma_{c \mapsto d}(x) \coloneqq x + (d - c), \qquad \sigma_{d \mapsto c}(x) \coloneqq x + (c - d).$$

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Subvarieties. In general, swappability passes to subvarieties.

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So. Every variety of MV-algebras, e.g. $\mathrm{V}\big(\{0, \frac{1}{2}, 1\}\big)$, has swappable constants.

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Heyting algebras. Heyting algebras do not have swappable constants:

$$\mathsf{HA} \not\models \neg\neg x = x.$$

Fibers over 0 and 1 of a homomorphism onto $\{0, 1\}$ can have different cardinalities.

Generating c-idempotent operations

Assume $\mathcal{V}$ has swappable constants, and fix $c \in \mathbf{I}$. For a primitive operation τ and constants $d_1, \ldots, d_n \in \mathbf{I}$, set

$$\widehat{\tau}_{d_1,\ldots,d_n}(x_1, \ldots, x_n) \coloneqq \sigma_{\tau(d_1,\ldots,d_n)\mapsto c}\Big(\tau\big(\sigma_{c\mapsto d_1}(x_1), \ldots, \sigma_{c\mapsto d_n}(x_n)\big)\Big).$$

Generation criterion

Each $\widehat{\tau}_{d_1,\ldots,d_n}$ is c-idempotent.
Moreover, these shifted primitive operations generate the clone of c-idempotent terms.

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Example: Boolean algebras, $c = 0$. Since 0 and 1 are swapped by $\neg$, from $\vee$ we get $\vee, \setminus, \setminus^{\mathrm{op}}, \wedge$:

$$\begin{aligned} x \mathbin{\widehat{\vee}_{0,0}} y &= x \vee y && \vee \\ x \mathbin{\widehat{\vee}_{1,0}} y &= \neg(\neg x \vee y) = x - y && \setminus \\ x \mathbin{\widehat{\vee}_{0,1}} y &= \neg(x \vee \neg y) = y - x && \setminus^{\mathrm{op}} \\ x \mathbin{\widehat{\vee}_{1,1}} y &= \neg(\neg x \vee \neg y) = x \wedge y && \wedge. \end{aligned}$$

Axioms for $c\text{-}\mathrm{Idl}(\mathcal{V})$

Now we turn the generating operations into an equational presentation.

Suppose $\mathcal{V}$ is presented by primitive operations and equations.

We want an equational presentation for the *subreducts* obtained by keeping the c-idempotent shifted operations.

Under swappability, this variety $c\text{-}\mathrm{Idl}(\mathcal{V})$ is presented by:

1. one basic operation for each shifted primitive operation;
2. the original axioms, rewritten in these shifted operations;
3. some bookkeeping equations.

Finite case

If $\mathcal{V}$ has a finite equational basis and $\mathbf{I}$ is finite, then $c\text{-}\mathrm{Idl}(\mathcal{V})$ has a finite equational basis.

Under swappability, from $A \in c\text{-}\mathrm{Idl}(\mathcal{V})$ we can build a $\mathcal{V}$-algebra

$$A \rtimes \mathbf{I} \qquad \text{with universe } A \times \mathbf{I}.$$

For an n-ary primitive operation τ and $d_i \in \mathbf{I}$, set

$$\tau^{A \rtimes \mathbf{I}}\big((a_1, d_1), \ldots, (a_n, d_n)\big) = \big(\widehat{\tau}^{A}_{d_1,\ldots,d_n}(a_1, \ldots, a_n), \tau^{\mathbf{I}}(d_1, \ldots, d_n)\big)\,.$$

Second coordinate in $\mathbf{I}$; first coordinate by shifted operations.

The semidirect-product-like construction

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Rings.

For a rng A, $A \rtimes \mathbb{Z}$ has universe $A \times \mathbb{Z}$.

For multiplication:

$$a \,\widehat{\cdot}_{m,n}\, b = (a+m)(b+n) - mn = ab + na + mb.$$

$$(a,m)\cdot(b,n) = \left(a \,\widehat{\cdot}^{A}_{m,n}\, b, mn\right) = (ab + na + mb, mn).$$

The semidirect-product-like construction

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Generalized Boolean algebras.

For $B \in \mathrm{GBA}$, $B \rtimes \mathbf{2}$ has universe $B \times \mathbf{2}$.

For join:

$$(a, d) \vee (b, e) = \left(a \mathbin{\widehat{\vee}^{B}_{d,e}} b,\ d \vee e\right), \qquad d, e \in \mathbf{2}.$$

$$\begin{array}{ll} (0,0) \mapsto a \vee b & (1,0) \mapsto a \setminus b \\ (0,1) \mapsto b \setminus a & (1,1) \mapsto a \wedge b \end{array}$$

The slice comparison

$$\boxed{\mathcal{V}/\mathbf{I} \simeq c\text{-}\mathrm{Idl}(\mathcal{V})}$$

$$(f\colon A \to \mathbf{I}) \longmapsto f^{-1}[\{c\}]$$

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$$(f \colon A \to \mathbf{I}) \longmapsto f^{-1}[\{c\}]$$
$$(B \rtimes \mathbf{I} \xrightarrow{\pi_2} \mathbf{I}) \longleftarrow B$$

Conclusion

Fix a variety $\mathcal{V}$ with swappable constants, and choose $c \in \mathbf{I}$.

We get: a concrete signature and equations for a variety of "*c-ideals of $\mathcal{V}$*":

$$c\text{-}\mathrm{Idl}(\mathcal{V})$$

This equals the class of subreducts of $\mathcal{V}$ with respect to the clone of c-idempotent terms, as well as

$$\mathcal{V}/\mathbf{I}.$$

Shifts allow to give a manageable presentation.

Also: semidirect product construction $B \mapsto B \rtimes \mathbf{I}$.

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Thank you!

Boolean toy example

Rewriting an axiom

The Boolean axiom

$$x \vee \neg x \approx 1$$

becomes

$$x \mathbin{\widehat{\vee}_{0,1}} \widehat{\neg}_0 x \approx \widehat{1}.$$

After simplification of the set of primitive operations, this can be read as

$$x - x \approx 0.$$

Boolean toy example

Bookkeeping

The shifted operation $x \mathbin{\widehat{\vee}_{1,0}} y$ was defined by the Boolean term

$$\neg(\neg x \vee y).$$

The bookkeeping equation says:

$$x \mathbin{\widehat{\vee}_{1,0}} y \approx \widehat{\neg}_1\big((\widehat{\neg}_0 x) \mathbin{\widehat{\vee}_{1,0}} y\big).$$