# Adding quantifiers via Stone duality


Marco Abbadini

Université catholique de Louvain, Belgium


Recording of the talk available at `https://youtu.be/uke8lU94Otc`

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Joint work in progress with Francesca Guffanti

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formulas modulo provable equivalence

$$\Downarrow$$

*Boolean algebras*

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Algebras of formulas $\xleftrightarrow{\mathrm{op}}$ Spaces of models.

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For example:

$$\text{free Boolean algebra on } X \quad \overset{\mathrm{op}}{\longleftrightarrow} \quad 2^X.$$

This is also reflected in coproducts.

| Boolean algebras | Stone spaces |
| --- | --- |
| coproducts | products |

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Various logical problems are algebraic questions about colimits (e.g.: interpolation / amalgamation).

There is a **first-order** analogue of this picture.

| Classical propositional logic | Classical first-order logic |
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| $\Downarrow$ | $\Downarrow$ |
| *Boolean algebras* | *Hyperdoctrines* |
| Stone spaces | Polyadic spaces |

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Again: algebras of formulas $\overset{\text{op}}{\longleftrightarrow}$ spaces of models, this time modulo elementary equivalence.

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Motivating example: in *logic on words*, coming from automata theory, a dual free construction was indeed developed; see the *Reutenauer-type theorem* in

Gehrke, Petrişan, Reggio (2017). Quantifiers on languages and codensity monads.

*Stone dual of Herbrand's theorem.*

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Provability of

**existential** *formulas*

in terms of provability of

**quantifier-free** *formulas.*

For $\varphi(x)$ a quantifier-free formula,

$$\vdash \exists x\, \varphi(x) \iff \text{there are ground terms } c_1, \ldots, c_n \text{ such that}$$
$$\vdash \varphi(c_1) \vee \cdots \vee \varphi(c_n).$$

For $\varphi(x)$ a quantifier-free formula,

$$\mathcal{T} \vdash \exists x\, \varphi(x) \iff \text{there are ground terms } c_1, \ldots, c_n \text{ such that } \mathcal{T} \vdash \varphi(c_1) \vee \cdots \vee \varphi(c_n).$$

It holds relative to any universal theory $\mathcal{T}$ (i.e., axiomatized by universal closures of quantifier-free formulas).

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First step towards a dual description of free hyperdoctrines.

For the talk: only relation symbols, no equality.

Fix a universal theory $\mathcal{T}$. (E.g.: the theory of preorders, the theory of directed graphs.)

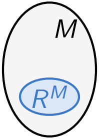
For two models $M, M'$ of $\mathcal{T}$, we write

$$M \equiv_{\exists} M' \iff M \text{ and } M' \text{ satisfy the same existential sentences.}$$

Example:

- $\mathcal{L} := \{R\}$, $R$ unary;
- $\mathcal{T} := \varnothing$.

Models of $\mathcal{T}$:



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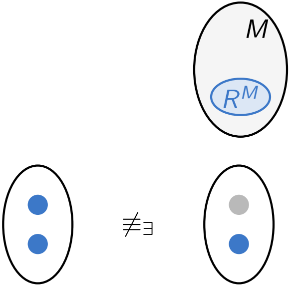

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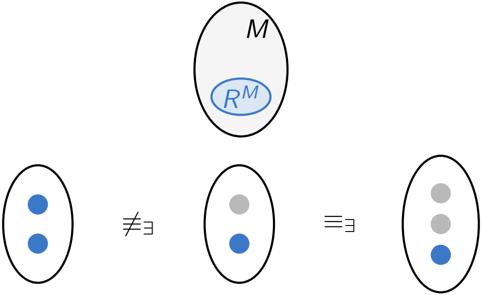
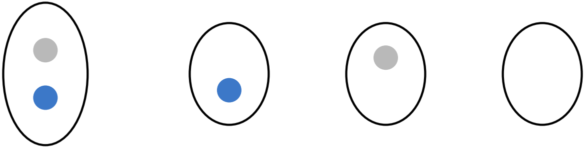


Models up to $\equiv_{\exists}$:



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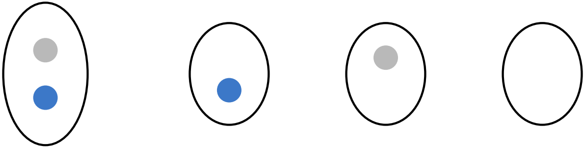

We will express this set in terms of the space of models up to *quantifier-free* equivalence.

For $X$ a finite set, an *$X$-pointed model* of $\mathcal{T}$ is

$$(\underset{\substack{\uparrow \\ \text{model of } \mathcal{T}}}{M}, \quad \underset{\substack{\uparrow \\ \text{function}}}{\mu} \colon X \to M).$$

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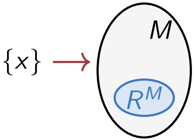
For $\mathcal{T} = \varnothing$ in the language $\{R \text{ (unary)}\}$, an $\{x\}$-pointed model is:



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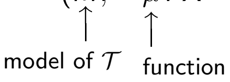
$$(M, \quad \mu\colon X \to M).$$

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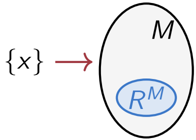
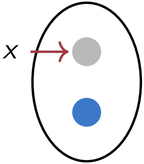
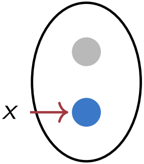
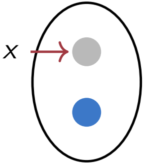



$(M, \mu) \equiv_{\text{q.f.}} (N, \nu) \iff$ they satisfy the same quantifier-free formulas with free variables in $X$.




$\not\equiv_{\text{q.f.}}$



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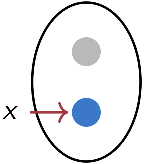


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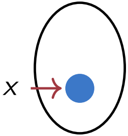

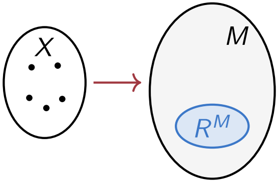
X
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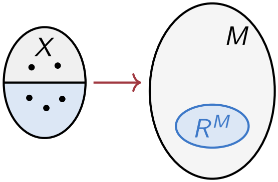
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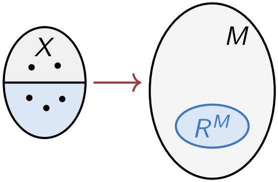



So, for this $\mathcal{T}$,

$$\frac{X\text{-pointed models of } \mathcal{T}}{\equiv_{\text{q.f.}}} \overset{1:1}{\longleftrightarrow} \mathcal{P}(X).$$

$$\begin{aligned} \mathsf{FinSet}^{\mathrm{op}} &\longrightarrow \mathsf{Stone} \\ X &\longmapsto \frac{\{X\text{-pointed models of } \mathcal{T}\}}{\equiv_{\mathrm{q.f.}}} \end{aligned}$$

For $\mathcal{T} = \varnothing$ in the language $\{R \text{ (unary)}\}$,

$$\frac{\{\text{pointed models of } \mathcal{T}\}}{\equiv_{\mathrm{q.f.}}} \cong \mathcal{P} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}.$$

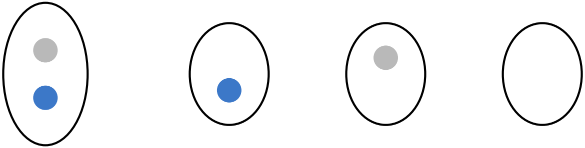
Our main contribution (Stone dual of Herbrand's theorem): for a universal theory $\mathcal{T}$, we describe the set (in fact, the Stone space)

$$\frac{\{\text{models of } \mathcal{T}\}}{\equiv_{\exists}}$$

in terms of the functor

$$\frac{\{\text{pointed models of } \mathcal{T}\}}{\equiv_{\mathrm{q.f.}}} : \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}.$$

For example, for $\mathcal{T} = \varnothing$ in the language $\{R \text{ (unary)}\}$, the set



should arise from the functor

$$\frac{\{\text{pointed models of } \mathcal{T}\}}{\equiv_{\text{q.f.}}} \cong \mathcal{P} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}.$$

## Theorem (The Stone dual of Herbrand's theorem)

*For a universal theory $\mathcal{T}$ (in a relational 1-sorted language without equality), the set*

$$\frac{\{\textit{models of } \mathcal{T}\}}{\equiv_{\exists}}$$

*is in bijection with the set of subfunctors of*

$$\frac{\{\textit{pointed models of } \mathcal{T}\}}{\equiv_{\mathrm{q.f.}}} : \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone}$$

*that map* **finite products** *of* $\mathsf{FinSet}^{\mathrm{op}}$ *to* **quasi-products** *of* Stone.

*Quasi-product* := the morphism to the product is surjective.

Concretely, a subfunctor of

$$\mathcal{P} \colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{Stone},$$

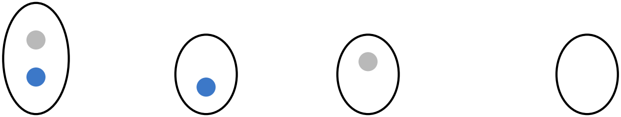
mapping finite products to quasi-products amounts to giving, for each $X \in \mathsf{FinSet}$, a subset

$$S_X \subseteq \mathcal{P}(X),$$

so that

1. (preimages) for $f \colon X \to Y$ and $A \in S_Y$, $f^{-1}[A] \in S_X$;
2. (disjoint unions) for $A \in S_X$ and $B \in S_Y$, $A \sqcup B \in S_{X \sqcup Y}$;
3. (empty set) $\varnothing \in S_\varnothing$.

$S_X = \mathcal{P}(X)$ $\qquad$ $S_X = \{X\}$ $\qquad$ $S_X = \{\varnothing\}$ $\qquad$ $S_X = \begin{cases} \{\varnothing\} & X = \varnothing, \\ \varnothing & X \neq \varnothing \end{cases}$



# In conclusion

Stone dual of Herbrand's theorem: for a universal theory $\mathcal{T}$, we describe the

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- We also allow function symbols and multiple sorts.

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We dualized the notion of "universal ultrafilter", used in our hyperdoctrinal version of Herbrand's theorem:

Abbadini, Guffanti. Freely adding one layer of quantifiers to a Boolean doctrine. arXiv.

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Obrigado!