On the symmetry behind duality

Marco Abbadini

Université catholique de Louvain, Belgium

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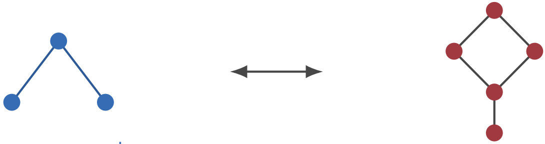
Stone duality for bounded distributive lattices

Spectral spaces $\overset{\text{op}}{\longleftrightarrow}$ *bounded distributive lattices*

$X \longmapsto$ compact opens of X



Opens = upsets

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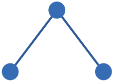

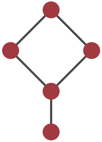

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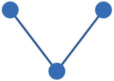
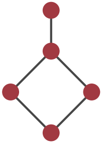




Opens = upsets

For a spectral space X, the *de Groot dual* X^∂ ("*inverse topology*"):

- X^∂ has the same underlying set as X,
- opens of X^∂ = complements of compact saturated sets of X.

It is still spectral.

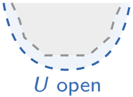
Compact saturated sets of X^∂ = complements of opens of X. So, $(X^\partial)^\partial = X$.

$$\text{open} \overset{\text{symmetry}}{\longleftrightarrow} \text{compact saturated}$$

$$\text{open} \overset{\text{symmetry}}{\longleftrightarrow} \text{compact saturated}$$

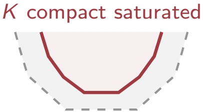
Every open U is the union of the compact open (hence saturated) sets it contains:

$$U = \bigcup_{\substack{C \text{ compact open} \\ C \subseteq U}} C.$$



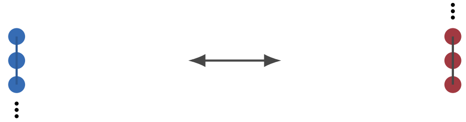
Every compact saturated K is the intersection of the compact open sets in which it is contained:

$$K = \bigcap_{\substack{C \text{ compact open} \\ K \subseteq C}} C.$$



Generalized spectral spaces $\overset{\text{op}}{\longleftrightarrow}$ *distributive lattices with* 0

$X \longmapsto$ compact opens of X



Stone 1937, Rump & Yang 2009, Wehrung 2017

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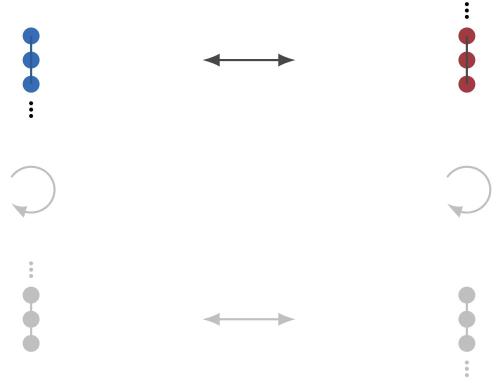


The set of complements of compact saturated sets of  is not a topology.

Stone 1937, Rump & Yang 2009, Wehrung 2017

Idea: to generalize it to a duality for (not necessarily bounded) distributive lattices.

$$\begin{array}{ccc} \textit{???} & \overset{\text{op}}{\longleftrightarrow} & \textit{distributive lattices} \\ X & \longmapsto & \text{compact opens of } X \end{array}$$

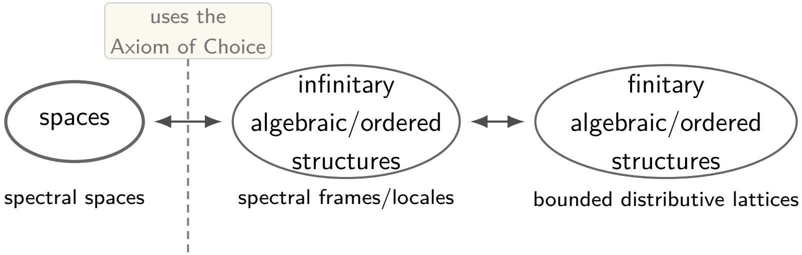
Difficulty to be overcome: in a topological space, $\varnothing$ is always compact open.

Idea: to weaken the notion of space: $\varnothing$ may be non-open, just like the whole space may be non-compact.

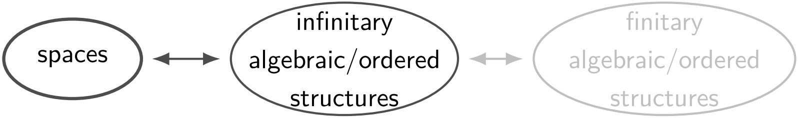
Goal: self-dual framework that extends some categorical dualities in the literature.

Idea: relax a bit the closure properties of the family of "opens".

Spatial and algebraic/ordered structures



Spatial and algebraic/ordered structures



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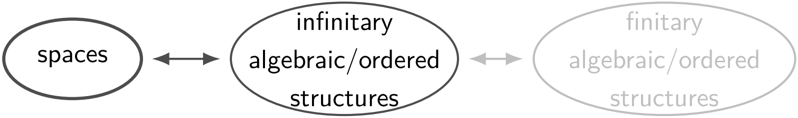

sober spaces **spatial frames/locales**

The class of sober spaces has only a partial symmetry.

In sober spaces:

Open sets	*Compact saturated sets*	
finite intersection of opens is open	finite union of compact satur. sets is compact satur.	✓
binary union of opens is open	binary intersect. of compact sat. sets may not be comp.	×
$\varnothing$ is open	X may not be compact	×
directed union of opens is open	down-directed intersection of compact satur. sets is compact satur.	✓

Goal: to obtain a duality...

- ... that generalizes the one between sober spaces and spatial frames, and
- ... in which both classes have an internal symmetry.

We take openness and compact saturatedness as *primitive* notions.

Even for Hausdorff spaces, compact saturated sets do not determine the open sets: on a countable set, I can put two different Hausdorff topologies with the same compact saturated sets.

Definition

A *ko-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}, \mathcal{O}$ sets of upsets of X s.t.

Sober spaces. Let $(X, \mathcal{O})$ be sober. Then $(X, \mathcal{K}, \mathcal{O})$ is a ko-space by setting

$$\leq = \text{specialization order}, \quad \mathcal{K} = \{\text{compact saturated sets}\}, \quad \mathcal{O} = \{\text{open sets}\}.$$

Definition

A *ko-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}, \mathcal{O}$ sets of upsets of X s.t.

1. $\mathcal{O}$ is closed under directed unions;
   $\mathcal{K}$ is closed under down-directed intersections.

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1. $\mathcal{O}$ is closed under directed unions;
   $\mathcal{K}$ is closed under down-directed intersections.
2. For $K \in \mathcal{K}$ and $\{U_i\}_i \subseteq \mathcal{O}$ directed
   $$K \subseteq \bigcup_i^{\uparrow} U_i \Rightarrow \exists i \; K \subseteq U_i$$
   For $U \in \mathcal{O}$ and $\{K_i\}_i \subseteq \mathcal{K}$ down-directed
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3. "Principal upsets are compact, principal downsets are closed": for $x \in X$,
   $$\uparrow x \in \mathcal{K}, \qquad X \setminus \downarrow x \in \mathcal{O}.$$

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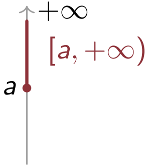
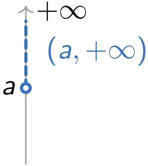
Another example

On $\mathbb{R}$ with its usual order, one can take

$$\mathcal{K} = \{\, [a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\varnothing\} \qquad \mathcal{O} = \{\, (a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\mathbb{R}\}$$





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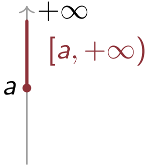

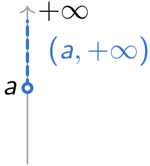

2.

$$K \subseteq \bigcup\nolimits_i^{\uparrow} U_i \Rightarrow \exists i \ K \subseteq U_i, \qquad \bigcap\nolimits_i^{\downarrow} K_i \subseteq U \Rightarrow \exists i \ K_i \subseteq U.$$

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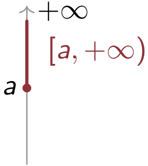
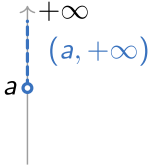
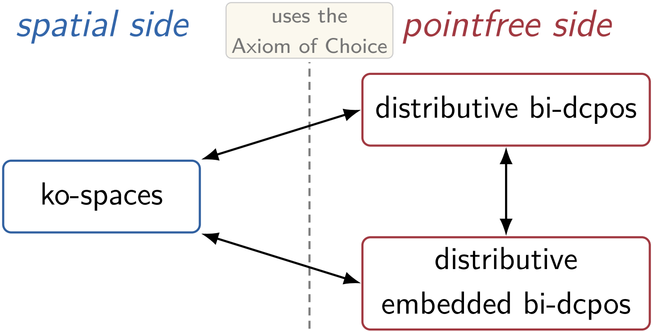
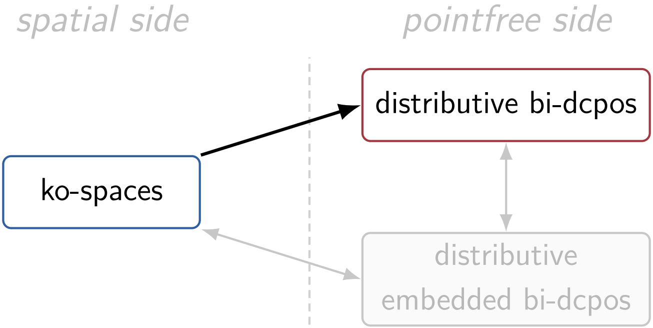




3.

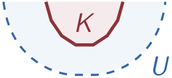
$$\uparrow x \in \mathcal{K}, \qquad \mathbb{R} \setminus \downarrow x \in \mathcal{O}.$$

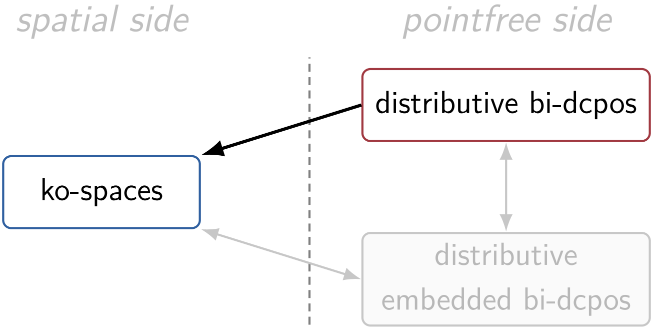

spatial side
uses the Axiom of Choice
pointfree side
ko-spaces
distributive bi-dcpos
distributive embedded bi-dcpos



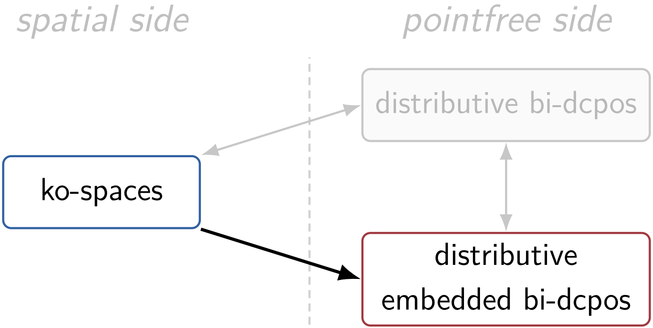
Forget the points, remember the two families and the inclusion relation:

$$(X,\ \mathcal{K},\ \mathcal{O}) \quad \longmapsto \quad (\mathcal{K},\ \mathcal{O},\ \subseteq : \mathcal{K} \looparrowright \mathcal{O}).$$



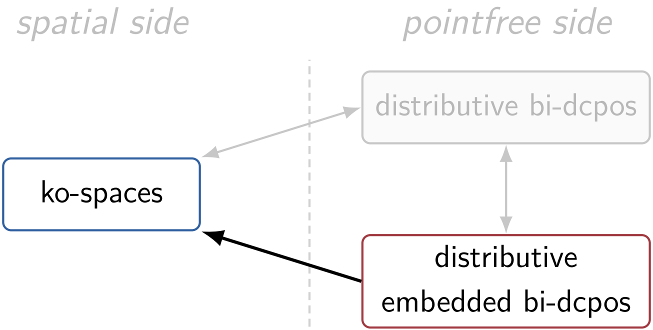


Take the set of completely prime pairs.



Remember the lattice of upsets of X and the two families, as subsets:

$$(X,\ \mathcal{K},\ \mathcal{O}) \quad\longmapsto\quad (\mathrm{Up}(X),\ \mathcal{K},\ \mathcal{O}).$$



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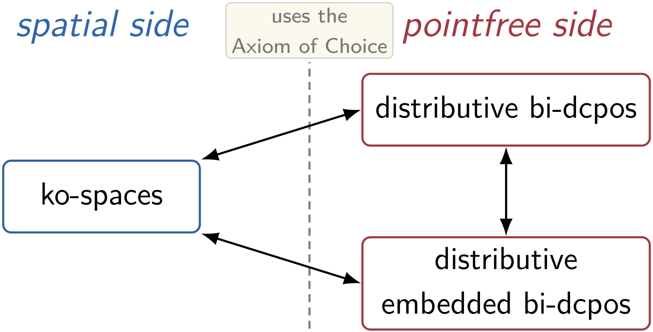
spatial side
uses the
Axiom of Choice
pointfree side
distributive bi-dcpos
ko-spaces
distributive
embedded bi-dcpos

An *embedded bi-dcpo* is a triple $(L, \mathcal{K}, \mathcal{O})$ such that:

- L is a complete lattice, $\mathcal{K}, \mathcal{O} \subseteq L$, and
  - every element of L is a join of elements from $\mathcal{K}$,
  - every element of L is a meet of elements from $\mathcal{O}$;
- $\mathcal{O}$ is closed under directed joins and $\mathcal{K}$ is closed under down-directed meets;
- for every $k \in \mathcal{K}$ and every directed family $(u_i)_i$ in $\mathcal{O}$,

  $$k \leq \bigvee\nolimits_i^{\uparrow} u_i \quad \Longrightarrow \quad \exists i \; k \leq u_i;$$

  for every $u \in \mathcal{O}$ and every down-directed family $(k_i)_i$ in $\mathcal{K}$,

  $$\bigwedge\nolimits_i^{\downarrow} k_i \leq u \quad \Longrightarrow \quad \exists i \; k_i \leq u.$$

It is *distributive* when L is distributive.

pointfree side

$$\begin{tikzcd}
\boxed{\text{distributive bi-dcpos}} \\
\boxed{\text{distributive embedded bi-dcpos}} \arrow[u]
\end{tikzcd}$$

Restrict the order of L to the two distinguished families:

$$(L,\ \mathcal{K},\ \mathcal{O}) \quad \longmapsto \quad (\mathcal{K},\ \mathcal{O},\ \leq\colon \mathcal{K} \looparrowright \mathcal{O}).$$

A *bi-dcpo* is a triple $(\mathcal{K}, \mathcal{O}, \lhd)$ where $\mathcal{O}$ is a dcpo, $\mathcal{K}$ is a co-dcpo, and $\lhd \colon \mathcal{K} \nrightarrow \mathcal{O}$ satisfies weakening and extensionality. Moreover:

- for every $k \in \mathcal{K}$ and every directed family $\{u_i\}_i \subseteq \mathcal{O}$,
$$k \lhd \bigvee_i^{\uparrow} u_i \quad \Longrightarrow \quad \exists i \ k \lhd u_i;$$
- for every $u \in \mathcal{O}$ and every down-directed family $\{k_i\}_{i \in I} \subseteq \mathcal{K}$,
$$\bigwedge_i^{\downarrow} k_i \lhd u \quad \Longrightarrow \quad \exists i \ k_i \lhd u.$$

Distributivity. A bi-dcpo is distributive when its concept lattice is a distributive lattice. Equivalently, it satisfies a two-sorted version of Gentzen's cut rule, schematically: for $k, \ell \in \mathcal{K}$ and $u, v \in \mathcal{O}$,

$$\frac{k \Rightarrow u, \ell \qquad \ell \lhd v \qquad v, k \Rightarrow u}{k \lhd u}.$$

Spatiality = distributivity = cut rule.

pointfree side

$$\begin{tikzcd} \boxed{\text{distributive bi-dcpos}} \arrow[d] \\ \boxed{\text{distributive embedded bi-dcpos}} \end{tikzcd}$$

Take the concept lattice:

$$(\mathcal{K},\ \mathcal{O},\ \lhd) \quad \longmapsto \quad \big(\mathbf{B}(\mathcal{K},\mathcal{O},\lhd),\ \mathcal{K},\ \mathcal{O}\big).$$

Main result

The category of ko-spaces, with appropriate morphisms, is dually equivalent to the category of *distributive bi-dcpos*, as well as to the category of *distributive embedded bi-dcpos*.

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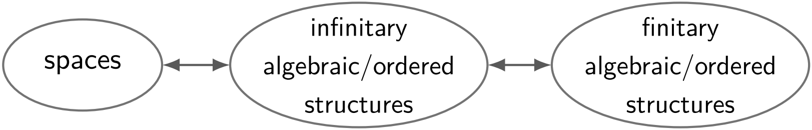
The category of ko-spaces, with appropriate morphisms, is dually equivalent to the category of *distributive bi-dcpos*, as well as to the category of *distributive embedded bi-dcpos*.

Balance:

- the structures are general enough to form self-symmetric classes including sober spaces and spatial frames,
- but also have enough axioms to obtain a duality that restricts to the one between sober spaces and spatial frames.



ko-spaces

distributive (embedded) bi-dcpos

$$\begin{tikzcd} \text{spaces} \arrow[r, leftrightarrow] & \text{infinitary algebraic/ordered structures} \arrow[r, leftrightarrow] & \text{finitary algebraic/ordered structures} \end{tikzcd}$$

ko-spaces

distributive (embedded) bi-dcpos

$$\begin{tikzcd} \begin{array}{c} X_A \\ = \mathrm{CP}(\mathbf{D}(A)) \end{array} & \begin{array}{c} \mathbf{D}(A) \\ = (\mathrm{Filt}(A), \mathrm{Idl}(A), \lhd) \end{array} \arrow[l] & A \arrow[l] \end{tikzcd}$$

distributive lattice
(not necessarily bounded)

Because of *local compactness*, it is enough to take $\mathcal{O}$ ($\mathcal{K}$ will be determined).

$$
\begin{tikzcd}
\text{sober spaces} \arrow[r, leftrightarrow] & \text{spatial frames} \\
\text{ko-spaces} \arrow[u, phantom, "\subseteq" sloped] \arrow[r, leftrightarrow] \arrow[loop below] & \text{distributive (embedded) bi-dcpos} \arrow[u, phantom, "\subseteq" sloped] \arrow[loop below]
\end{tikzcd}
$$

Preprint:

Abbadini and Jung,

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arXiv:2507.18245

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Thank you