On the symmetry between openness and compactness

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Based on joint work with Achim Jung
(*On the symmetry behind duality*, arXiv:2507.18245)

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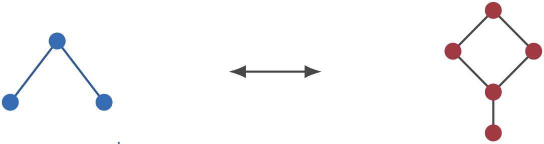
Recording of the talk: https://youtu.be/A2W2gxphThA

Stone duality for bounded distributive lattices

Spectral spaces $\overset{\text{op}}{\longleftrightarrow}$ *bounded distributive lattices*

$X \longmapsto$ compact opens of X



Opens = upsets

Stone duality for bounded distributive lattices

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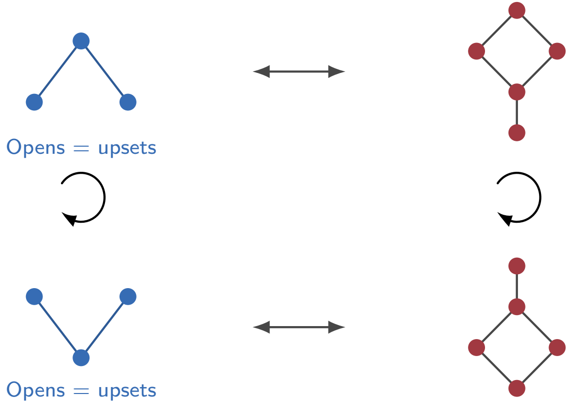
X	$\longmapsto$	compact opens of X



For a spectral space X, the *de Groot dual* X^∂:

- X^∂ has the same underlying set as X,
- opens of X^∂ = complements of compact saturated sets of X.

It is still spectral.

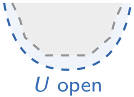
Compact saturated sets of X^∂ = complements of opens of X. So, $(X^\partial)^\partial = X$.

$$\text{open} \xleftrightarrow{\text{symmetry}} \text{compact saturated}$$

open $\overset{\text{symmetry}}{\longleftrightarrow}$ compact saturated

Every open U is the union of the compact open (hence saturated) sets it contains:

$$U = \bigcup_{\substack{C \text{ compact open} \\ C \subseteq U}} C.$$

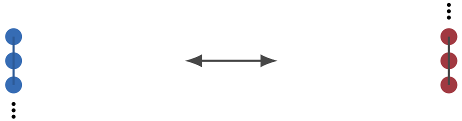


Every compact saturated K is the intersection of the compact open sets in which it is contained:

$$K = \bigcap_{\substack{C \text{ compact open} \\ K \subseteq C}} C.$$

Generalized spectral spaces $\xleftrightarrow{\text{op}}$ *distributive lattices with* 0

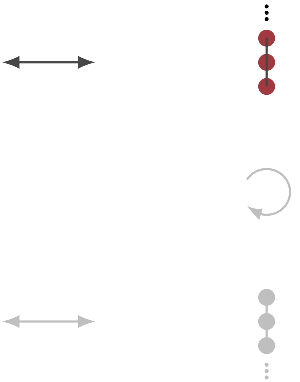
$X \longmapsto$ compact opens of X



Stone 1937, Rump & Yang 2009, Wehrung 2017

Generalized spectral spaces $\xleftrightarrow{\text{op}}$ *distributive lattices with 0*

$X \longmapsto$ compact opens of X



The set of complements of compact saturated sets of is not a topology.

Stone 1937, Rump & Yang 2009, Wehrung 2017

Idea: generalize this to a duality for (not necessarily bounded) distributive lattices.

$$\begin{array}{ccc} \textit{???} & \overset{\text{op}}{\longleftrightarrow} & \textit{distributive lattices} \\ X & \longmapsto & \text{compact opens of } X \end{array}$$

Difficulty to be overcome: in a topological space, $\varnothing$ is always compact open.

Idea: to weaken the notion of space: $\varnothing$ may be non-open, just like the whole space may be non-compact.

Goal: a self-dual framework that extends some categorical dualities in the literature.

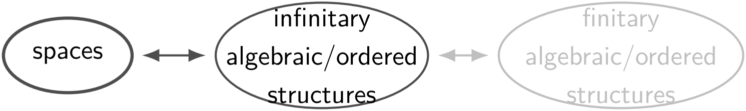
Idea: relax a bit the closure properties of the family of “opens”.

Spatial and algebraic/ordered structures

uses the Axiom of Choice

spaces	infinitary algebraic/ordered structures	finitary algebraic/ordered structures
spectral spaces	spectral frames/locales	bounded distributive latt.
stably compact sp.	stably compact frames/locales	proximity lattices
sober spaces	spatial frames/locales	
loc. comp. sob. sp.	distributive continuous lattices (i.e., continuous frames)	
	continuous domains	information systems
de Groot self-duality	*Lawson self-duality*	*order self-duality*

Spatial and algebraic/ordered structures



sober spaces **spatial frames/locales**

The class of sober spaces has only a partial symmetry.

In sober spaces:

Open sets	*Compact saturated sets*	
finite intersection of opens is open	finite union of compact satur. sets is compact satur.	✓
binary union of opens is open	binary intersect. of compact sat. sets may not be comp.	×
$\varnothing$ is open	X may not be compact	×
directed union of opens is open	codirected intersection of compact satur. sets is compact satur.	✓

Goal: to obtain a duality...

- ... that generalizes the one between sober spaces and spatial frames, and
- ... in which both classes have an internal symmetry: "de Groot"-like on the space side, and "Lawson"-like on the pointfree side.

We take openness and compact saturatedness as *primitive* notions.

Even for Hausdorff spaces, compact saturated sets do not determine the open sets: on a countable set, I can put two different Hausdorff topologies with the same compact saturated sets.

Definition

A *ko-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}, \mathcal{O}$ sets of upsets of X s.t.

Sober spaces. Let $(X, \mathcal{O})$ be sober. Then $(X, \mathcal{K}, \mathcal{O})$ is a ko-space by setting

$$\leq = \text{specialization order}, \quad \mathcal{K} = \{\text{compact saturated sets}\}, \quad \mathcal{O} = \{\text{open sets}\}.$$

Definition

A *ko-space* is a triple $(X, \mathcal{K}, \mathcal{O})$ with X a poset and $\mathcal{K}, \mathcal{O}$ sets of upsets of X s.t.

1. $\mathcal{O}$ is closed under directed unions;
   $\mathcal{K}$ is closed under codirected intersections.

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1. $\mathcal{O}$ is closed under directed unions;
   $\mathcal{K}$ is closed under codirected intersections.
2. For $K \in \mathcal{K}$ and $\{U_i\}_i \subseteq \mathcal{O}$ directed
   $$K \subseteq \bigcup_i^{\uparrow} U_i \Rightarrow \exists i \; K \subseteq U_i$$
   For $U \in \mathcal{O}$ and $\{K_i\}_i \subseteq \mathcal{K}$ codirected
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   $$\bigcap\nolimits_i^{\downarrow} K_i \subseteq U \Rightarrow \exists i\ K_i \subseteq U$$
3. "Principal upsets are compact, principal downsets are closed": for $x \in X$,
   $$\uparrow x \in \mathcal{K}, \qquad X \setminus \downarrow x \in \mathcal{O}.$$

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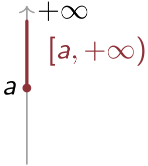
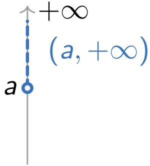
Another example

On $\mathbb{R}$ with its usual order, one can take

$$\mathcal{K} = \{\, [a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\varnothing\} \qquad \mathcal{O} = \{\, (a, +\infty) \mid a \in \mathbb{R} \,\} \cup \{\mathbb{R}\}$$





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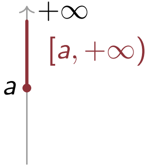

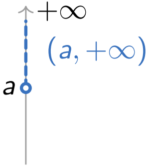

2.

$$K \subseteq \bigcup_i^{\uparrow} U_i \Rightarrow \exists i \; K \subseteq U_i, \qquad \bigcap_i^{\downarrow} K_i \subseteq U \Rightarrow \exists i \; K_i \subseteq U.$$

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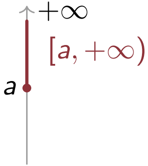
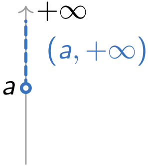
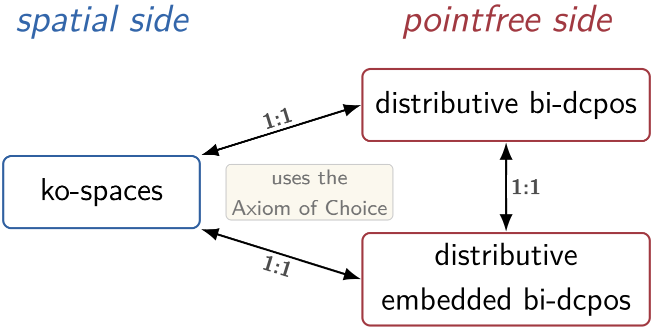
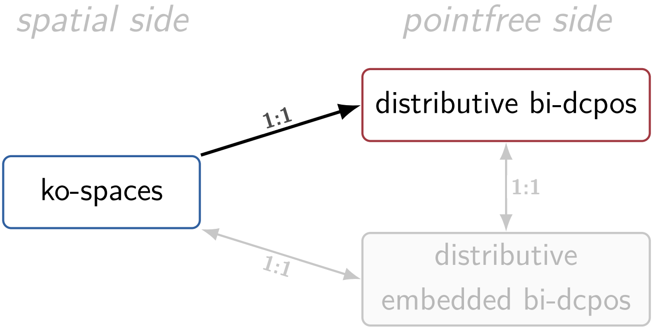




3.

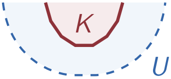
$$\uparrow x \in \mathcal{K}, \qquad \mathbb{R} \setminus \downarrow x \in \mathcal{O}.$$

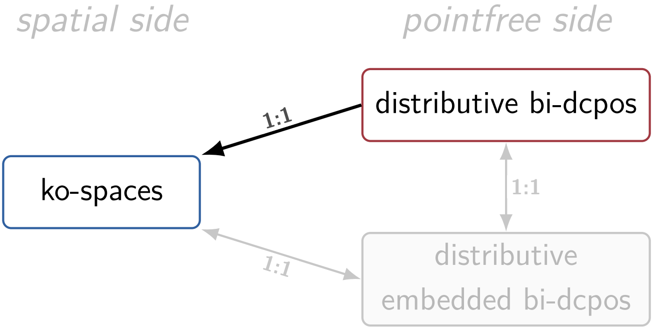

spatial side
pointfree side
distributive bi-dcpos
1:1
ko-spaces
uses the
Axiom of Choice
1:1
1:1
distributive
embedded bi-dcpos



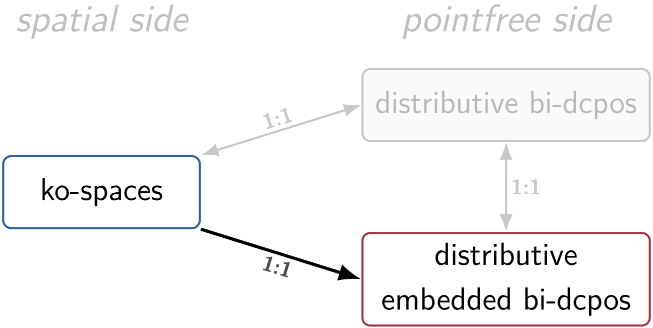
Forget the points, remember the two families, each with its inclusion order, and the inclusion relation between the two:

$$(X,\ \mathcal{K},\ \mathcal{O}) \quad \longmapsto \quad (\mathcal{K},\ \mathcal{O},\ \subseteq : \mathcal{K} \nrightarrow \mathcal{O}).$$



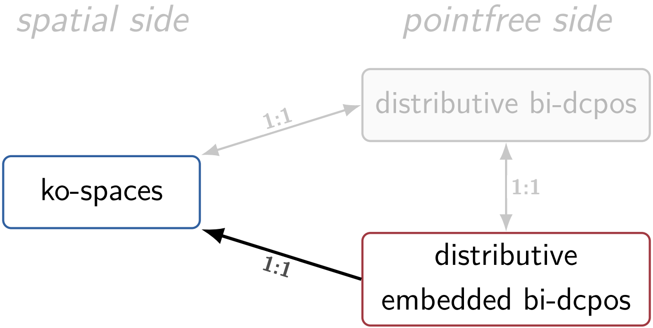


Take the set of completely prime pairs.



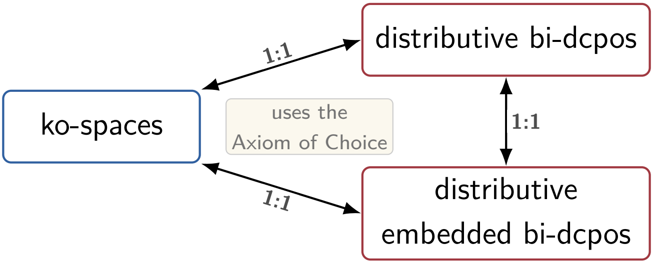
Remember the lattice of upsets of X and the two families, as subsets:

$$(X,\ \mathcal{K},\ \mathcal{O}) \quad \longmapsto \quad (\mathrm{Up}(X),\ \mathcal{K},\ \mathcal{O}).$$



Take the set of completely prime pairs.

spatial side *pointfree side*



Definition

An *embedded bi-dcpo* is a triple $(L, \mathcal{K}, \mathcal{O})$ where L is a complete lattice and $\mathcal{K}, \mathcal{O} \subseteq L$ are such that:

- $\mathcal{O}$ is closed under directed sups and $\mathcal{K}$ under codirected infs;
- *Order-density*:
  - every element of L is a sup of elements of $\mathcal{K}$,
  - every element of L is an inf of elements of $\mathcal{O}$;
- *Double compactness*:

$$k \leq \bigvee_i^{\uparrow} u_i \implies \exists i\ k \leq u_i; \qquad \bigwedge_i^{\downarrow} k_i \leq u \implies \exists i\ k_i \leq u.$$

It is *distributive* when L is distributive.

Distributivity $\iff$ spatiality.

pointfree side

$$\boxed{\text{distributive embedded bi-dcpos}} \xrightarrow{\text{restrict the order}} \boxed{\text{distributive bi-dcpos}}$$

$$(L, \mathcal{K}, \mathcal{O}) \longmapsto (\mathcal{K}, \mathcal{O}, \leq : \mathcal{K} \nrightarrow \mathcal{O})$$

Embedded bi-dcpos $\overset{1:1}{\longleftrightarrow}$ bi-dcpos

Definition

Embedded bi-dcpo: $(L, \mathcal{K}, \mathcal{O})$: L compl.,

- $\mathcal{O}$ closed under dir. sup, $\mathcal{K}$ codir. inf;
- *Order-density*:

  every $a \in L$ is a sup of elements of $\mathcal{K}$,

  every $a \in L$ is an inf of elements of $\mathcal{O}$;
- *Double compactness*:

  $k \leq \bigvee_i^{\uparrow} u_i \Longrightarrow \exists i\ (k \leq u_i)$,

  $\bigwedge_i^{\downarrow} k_i \leq u \Longrightarrow \exists i\ (k_i \leq u)$.

It is *distributive* if the lattice L is distributive; equivalently, if, for all $a, b, c \in L$,

$$\frac{a \leq b \vee c \quad c \wedge a \leq b}{a \leq b}\ (\mathrm{Cut})$$

Definition

Bi-dcpo: $(\mathcal{K}, \mathcal{O}, \lhd)$, where

- $\mathcal{O}$ is a dcpo, $\mathcal{K}$ a co-dcpo;
- *Extensionality*:

  $u \leq v \Longleftrightarrow \forall k \in \mathcal{K}\, (k \lhd u \Rightarrow k \lhd v)$,

  $k \leq l \Longleftrightarrow \forall u \in \mathcal{O}\, (l \lhd u \Rightarrow k \lhd u)$;
- *Double compactness*:

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  $\bigwedge_i^{\downarrow} k_i \lhd u \Longrightarrow \exists i\ (k_i \lhd u)$.

It is *distributive* if

$$\frac{\text{"}k \leq u \vee \ell\text{"} \quad \text{"}\ell \leq v\text{"} \quad \text{"}v \wedge k \leq u\text{"}}{\text{"}k \leq u\text{"}}$$

(see the next slide for the precise defin.).

$$\frac{\forall w \in \mathcal{O}\left(\begin{cases} u \leq w \\ \ell \lhd w \end{cases} \Rightarrow k \lhd w\right) \qquad \ell \lhd v \qquad \forall h \in \mathcal{K}\left(\begin{cases} h \leq k \\ h \lhd v \end{cases} \Rightarrow h \lhd u\right)}{k \lhd u}.$$

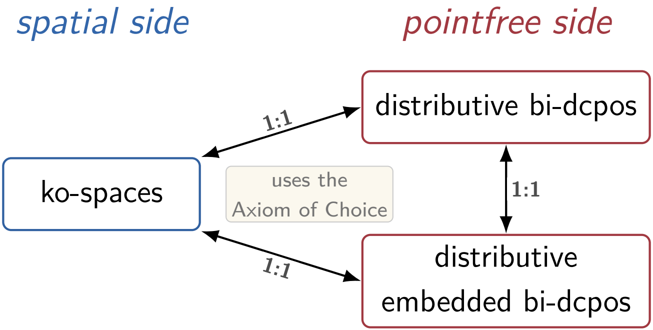
pointfree side

$$
\begin{tikzcd}
\boxed{\text{distributive bi-dcpos}} \arrow[d] \\
\boxed{\text{distributive embedded bi-dcpos}}
\end{tikzcd}
$$

Take the concept lattice of $(\mathcal{K}, \mathcal{O}, \lhd)$. It is a complete lattice in which $\mathcal{K}$ and $\mathcal{O}$ embed.

Main result

The category of *ko-spaces*, with appropriate morphisms, is dually equivalent to the category of *distributive bi-dcpos*, as well as to the category of *distributive embedded bi-dcpos*.



$$\begin{tikzcd}
\text{sober spaces} \arrow[r, leftrightarrow] & \text{spatial frames} \\
\text{ko-spaces} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow] \arrow[loop below, "\text{de Groot self-duality}"] & \text{distributive (embedded) bi-dcpos} \arrow[u, phantom, sloped, "\subseteq"] \arrow[loop below, "\text{Lawson self-duality}"]
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\end{tikzcd}$$

Balance:

- the structures are general enough to form self-symmetric classes including sober spaces and spatial frames,
- but also have enough axioms to obtain a duality that restricts to the one between sober spaces and spatial frames.

spaces $\longleftrightarrow$ infinitary algebraic/ordered structures $\longleftrightarrow$ finitary algebraic/ordered structures

ko-spaces

distributive (embedded) bi-dcpos

spaces ⟷ infinitary algebraic/ordered structures ⟷ finitary algebraic/ordered structures

ko-spaces

distributive (embedded) bi-dcpos

$$\begin{array}{c} X_A \\ = \mathrm{CP}(\mathbf{D}(A)) \end{array} \longleftarrow \begin{array}{c} \mathbf{D}(A) \\ = (\mathrm{Filt}(A), \mathrm{Idl}(A), \lhd) \end{array} \longleftarrow A$$

distributive lattice
(not necessarily bounded)

Because of *local compactness* (more precisely: *bicontinuity*), it is enough to take $\mathcal{O}$ (as $\mathcal{K}$ will be determined).

Bicontinuity

Bicontinuity (essentially: local compactness):

- Allows the two families $\mathcal{K}$ and $\mathcal{O}$ to determine each other (as Lawson duals);
- Allows us to pass to the finitary world without losing information (via "information systems").

Continuous domains as bi-dcpos

Continuous domains can be identified with a class of bi-dcpos: the *bicontinuous bi-dcpos*.

(They are not necessarily distributive.)

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Given a continuous domain D, consider

$$\mathcal{O} \coloneqq D, \qquad \mathcal{K} \coloneqq \mathrm{ScottOpFilt}(D) \quad \text{ordered by reverse inclusion,}$$

and set

$$F \lhd u \quad \Longleftrightarrow \quad F \ni u.$$

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Then

$$\big(\mathrm{ScottOpFilt}(D), D, \ni\big)$$

is a bi-dcpo.

Why this is a bi-dcpo

- $\mathcal{O} = D$ is a dcpo, and $\mathcal{K} = \mathrm{ScottOpFilt}(D)$, ordered by reverse inclusion, is a co-dcpo (a directed union of Scott-open filters is a Scott-open filter).

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- *Extensionality*:

$$G \leq_{\mathcal{K}} F \Leftrightarrow G \supseteq F \Leftrightarrow (\forall u\ u \in F \Rightarrow u \in G) \Leftrightarrow (\forall u\ F \lhd u \Rightarrow G \lhd u)$$

$$u \leq v \Leftrightarrow (\forall F \in \mathcal{K}\ u \in F \Rightarrow v \in F) \Leftrightarrow (\forall F \in \mathcal{K}\ F \lhd u \Rightarrow F \lhd v)$$

filters are determined by their elements, and D is open-filter-determined.

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  filters are determined by their elements, and D is open-filter-determined.
- *Double compactness*: for $(u_i)_i$ directed in D,

  $$F \lhd \bigvee_i^{\uparrow} u_i \iff \bigvee_i^{\uparrow} u_i \in F \iff \exists i \, (F \lhd u_i);$$

  for $(F_i)_i$ codirected in $\mathcal{K}$,

  $$\bigwedge_i^{\downarrow} F_i \lhd u \iff u \in \bigcup_i^{\uparrow} F_i \iff \exists i \, (F_i \lhd u).$$

Bicontinuity

Definition

Embedded bi-dcpo: $(L, \mathcal{K}, \mathcal{O})$: L compl.,

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It is *bicontinuous* if

- *Local compactness*: for $k \in \mathcal{K}$, $u \in \mathcal{O}$, if $k \leq u$, then $\exists u' \in \mathcal{O}\ \exists k' \in \mathcal{K}$ such that

$$k \leq u' \leq k' \leq u;$$

- for $u \in \mathcal{O}$, $\{k \in \mathcal{K} \mid k \leq u\}$ is directed;
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$$k \lhd u' \blacktriangleleft k' \lhd u;$$

- for $u \in \mathcal{O}$, $\{k \in \mathcal{K} \mid k \lhd u\}$ is directed;
- for $k \in \mathcal{K}$, $\{u \in \mathcal{O} \mid k \lhd u\}$ is codirect.

Here, $u \blacktriangleleft k$ means

$\forall l \in \mathcal{K}\, \forall v \in \mathcal{O}\,(l \lhd u \text{ and } k \lhd v \Rightarrow l \lhd v)$.

A two-sorted presentation of continuous domains

continuous domains

bicontinuous bi-dcpos

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$$\text{\textit{continuous domains}} \xrightarrow{D \longmapsto (\mathrm{ScottOpFilt}(D),\, D,\, \ni)} \text{\textit{bicontinuous bi-dcpos}}$$

$$\mathcal{O} = D$$

$$\mathcal{K} = \mathrm{ScottOpFilt}(D) \quad \text{is ordered by reverse inclusion,}$$

$$F \lhd u \iff u \in F.$$

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$$\begin{tikzcd}[column sep=huge]
\textit{continuous domains} \arrow[r, shift left, "D \longmapsto (\mathrm{ScottOpFilt}(D),\, D,\, \ni)"] & \textit{bicontinuous bi-dcpos} \arrow[l, shift left, "\mathcal{O} \longleftarrow\!\shortmid (\mathcal{K},\, \mathcal{O},\, \lhd)", "1:1"']
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Conversely, if $(\mathcal{K}, \mathcal{O}, \lhd)$ is bicontinuous, then $\mathcal{O}$ is a continuous domain.

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This correspondence uses the axiom of dependent choice.

Lawson duality is the swap of the two sorts

The object-level correspondence extends to morphisms:

continuous domains and open filter morphisms	$\overset{1:1}{\longleftrightarrow}$	bicontinuous bi-dcpos and Galois morphisms.

Theorem

The category of continuous domains and open filter morphisms is equivalent to the category of bicontinuous bi-dcpos and Galois morphisms.

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Each of the two categories has a natural self-duality.

- For continuous domains: Lawson self-duality.
- For bicontinuous bi-dcpos:

$$\boxed{(\mathcal{K}, \mathcal{O}, \lhd)} \xleftrightarrow{\text{swap the two sorts}} \boxed{(\mathcal{O}^{\mathrm{op}}, \mathcal{K}^{\mathrm{op}}, \rhd)}$$

$$u \rhd k \iff k \lhd u,$$

with the reversed orders on the two sorts.

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Under the continuous-domain presentation, this is precisely Lawson self-duality.

What bicontinuity buys us

Bicontinuity allows us to pass faithfully between the two-sorted and the one-sorted settings:

- If $(\mathcal{K}, \mathcal{O}, \lhd)$ is a bicontinuous bi-dcpo, then $\mathcal{K}$ and $\mathcal{O}$ determine each other (as Lawson duals).
- If $(\mathcal{K}, \mathcal{O}, \lhd)$ is a bicontinuous bi-dcpo, then, whenever $\mathcal{O}$ has binary meets and joins,
$$(\mathcal{K}, \mathcal{O}, \lhd) \text{ is distributive} \iff \mathcal{O} \text{ is distributive.}$$
Dually, the same holds with $\mathcal{K}$ in place of $\mathcal{O}$.

$$\textit{Bicontinuity} + \text{lattice distributivity of } \mathcal{O} \text{ or } \mathcal{K}$$
$$\Longrightarrow \text{spatiality.}$$

In particular, this gives another explanation of why every locally compact frame is spatial.

Recap

ko-spaces $\overset{\text{op}}{\longleftrightarrow}$ *distributive (embedded) bi-dcpos*

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bicontinuity $\Longrightarrow$ $\mathcal{K}$ and $\mathcal{O}$ are interdefinable ko-spaces $\xleftrightarrow{\text{op}}$ *distributive (embedded) bi-dcpos*

bicontinuity $\Longrightarrow$ $\mathcal{K}$ and $\mathcal{O}$ are interdefinable

bicontinuous (embedded) bi-dcpos $\xleftrightarrow{1:1}$ continuous domains

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Preprint:

Abbadini and Jung,

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arXiv:2507.18245

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Thank you