A categorical duality for metrically complete lattice-ordered Abelian groups with unit

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Recording of the talk available at `https://youtu.be/WvWTfdxNUhU`.

Based on:

M. Abbadini, V. Marra, L. Spada.
Stone-Gelfand duality for metrically complete lattice-ordered groups.
Advances in Mathematics, 2025.

Categorical Dualities

Categorical dualities are bridges between two worlds:

Geometry $\longleftrightarrow$ **Algebra**

Each side encodes the same information, but in a very different language.

Categorical dualities:

an improved version of representation theorems.

Representation theorems show how abstract algebraic structures can be realized concretely via some geometric/topological objects.

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Representation theorems show how abstract algebraic structures can be realized concretely via some geometric/topological objects.

Categorical dualities go further:

they make the correspondence **bijective**,
and let us use it **to its full power —**
translating problems, theorems, and intuitions across the duality.

A short journey through three categorical dualities

1930s **Stone duality:** connects topology and logic.

1940s **Yosida duality:** connects topology and analysis.

2025 **A new duality:** brings the two under one umbrella.

I. Stone duality

Logic becomes algebra: Boolean algebras

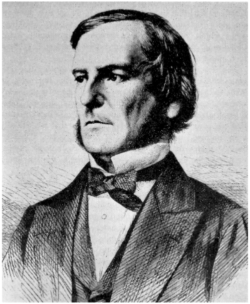
A *Boolean algebra* ($\approx$ Boole, 1847) is a set B (think of a set of propositions) equipped with operations

$$p \vee q, \quad p \wedge q, \quad \neg p, \quad 0, \quad 1$$

satisfying familiar laws:

$$p \wedge \neg p = 0, \dots$$

Example: the power set $\mathcal{P}(X)$ of a set X.



George Boole (1815–1864)

Stone's representation: Boolean algebras as fields of sets

For any set X, any set $\mathcal{F} \subseteq \mathcal{P}(X)$ that is closed under

$$A \cap B, \quad A \cup B, \quad A^c, \quad \varnothing, \quad X$$

(i.e., any *field of sets* over X) is a **Boolean algebra**.

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(Birkhoff-)Stone's representation theorem ((1933-)1936)

Every Boolean algebra is isomorphic to some field of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ over some set X.

From representation to duality

So far, we have seen a **representation theorem.**

the base

Stone proved more: a **categorical duality**, for which we need *two ingredients*:

- 1:1 corresp. on objects
- 1:1 corresp. on morph.

the ingredients

#1: Bijective correspondence on objects

For each Boolean algebra, there is a **canonical** representation: the unique one that is **separating** and **compact**.

Stone space :=
space with the following properties:
(i) the *clopens* (= *closed opens*) **separate** the elements,
(ii) it is **compact**.

Example: any finite discrete space.

Stone spaces $\xleftrightarrow{1:1}$ Boolean algebras.

Stone space $\longmapsto$ Boolean alg. of *clopen* subsets.

Example: finite discrete space $\longmapsto$ its power set.

a tomatopological sauce:
Stone sp. $\xleftrightarrow{1:1}$ Bool. alg.

#2: Bijective correspondence on maps, reversed

Boolean homomorphisms

$A \to B$

$$\Big\updownarrow \ 1{:}1$$

continuous functions

Stone space of B $\to$ Stone space of A.

mozzarphisms:

$$A \longrightarrow B$$

$$X_A \longleftarrow X_B$$

Example:

$$X \hookrightarrow Y$$

between finite discrete spaces

power set of Y $\twoheadrightarrow$ power set of X

$$A \longmapsto A \cap X$$

1. Bijection between Boolean algebras and Stone spaces
2. Bijection between the appropriate maps, but with opposite direction.

A *category* (Eilenberg, Mac Lane, 1945) consists of:

- a class "of structures" (e.g. Boolean algebras), and
- a class "of maps between them" (e.g. Boolean homomorphisms).

A *categorical duality* is a bijective correspondence between two categories, but reversing the direction of the maps.

Stone duality (1936)

There is a **categorical duality** between:

1. the category of **Boolean algebras** with Boolean homomorphisms, and
2. the category of **Stone spaces** with continuous functions.

Translating problems

Logic (interpolation). If $\varphi \vdash \psi$ in classical propositional logic, is there a formula θ, using only their common variables, such that

$$\varphi \vdash \theta \qquad \text{and} \qquad \theta \vdash \psi?$$

Algebra (amalgamation).
Boolean algebras.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & B \arrow[d, hook, dashed, "u"] \\ C \arrow[r, hook, dashed, "v"'] & D \end{tikzcd}$$

Topology.
Stone spaces.

$$\begin{tikzcd} P \arrow[r, two heads, dashed, "p"] \arrow[d, two heads, dashed, "q"'] & X \arrow[d, two heads, "f"] \\ Y \arrow[r, two heads, "g"'] & Z \end{tikzcd}$$

Stone duality: all three are the same question — and the answer is immediate on the topological side. This proves amalgamation, and hence interpolation.

II. Yosida duality

The late 1930s / early 1940s

After Stone's work, mathematicians realized a certain pattern: various structures can be represented as spaces of (real-valued / complex-valued) *continuous functions* on a compact Hausdorff space.

This gave rise to several representation theorems in the late '30s and early '40s.

Some names in the game: Gelfand, Kolmogorov, Naimark, Kakutani, Yosida, the Krein brothers, Stone.

Some structures in the game: commutative C^*-algebras, Banach lattices and Kakutani's (M)-spaces, vector lattices, divisible lattice-ordered Abelian groups.

Most famous one: Gelfand–Naimark duality between unital commutative C^*-algebras and compact Hausdorff spaces.

The most relevant for this talk is Yosida duality (1941):

Compact Hausdorff spaces

$\updownarrow$ duality

Metrically complete unital vector lattices

Metrically complete unital vector lattices are certain real Banach spaces with some more structure.

Continuous real functions

For a compact Hausdorff space X, the set $C(X, \mathbb{R})$ of real–valued continuous functions on X can be equipped with various pointwise defined operations:

- pointwise sum $(f, g) \mapsto f + g$;
- pointwise maximum $(f, g) \mapsto f \vee g$ and minimum $(f, g) \mapsto f \wedge g$;
- for any $\lambda \in \mathbb{R}$, the pointwise scalar multiplication by λ, i.e., $f \mapsto \lambda \cdot f$;
- constant functions 0 and 1.

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These operations are enough to recover the space X from $C(X, \mathbb{R})$ equipped with this structure.

For the following definition, think of $C(X, \mathbb{R})$.

Definition (Riesz, 1928)

A *vector lattice* is a real vector space equipped with a lattice order (i.e. a partial order with binary suprema and infima) s.t.:

1. (Translation invariance) if $x \leq y$ then $x + z \leq y + z$;
2. (Positive homogeneity) if $x \leq y$ then $\lambda x \leq \lambda y$ for any real $\lambda \geq 0$.

For Yosida's duality a bit more structure is needed.

Definition

A *metrically complete unital vector lattice* is a vector lattice V equipped with a designated element $1 \in V$ such that the following defines a complete metric on V:

$$\mathrm{dist}(v, w) := \inf \{\lambda \in \mathbb{R}_{\geq 0} \mid -\lambda 1 \leq v - w \leq \lambda 1\},$$

Yosida proved that every metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for some compact Hausdorff space X.

Yosida duality (1941)

The category of compact Hausdorff spaces is dually equivalent to the category of metrically complete unital vector lattices.

An inclusion of spaces $X \hookrightarrow Y$ gives a restriction function

$$\begin{aligned} C(Y, \mathbb{R}) &\twoheadrightarrow C(X, \mathbb{R}) \\ f &\longmapsto f|_X, \end{aligned}$$

that preserves $+$, $\vee$, $\wedge$, scalar multiplication, the unit 1 (hence is a morphism of metrically complete unital vector lattices).

III. An umbrella for both

Two dualities, two mirrors

So far, we have seen two categorical dualities:

- **Stone duality** between Stone spaces and Boolean algebras (very discrete structures),
- **Yosida duality** between compact Hausdorff spaces and metrically complete unital vector lattices (very continuous structures).

Every Stone space X is in particular compact Hausdorff:

1. Stone associates to it the Boolean algebra of clopens of X, i.e., $C(X, \{0, 1\})$;
2. Yosida associates to it $C(X, \mathbb{R})$.

We are using two different dualities, obtaining for the same space two very different objects.

1. We recall a class of algebraic structures that can specialize to
   - something similar to vector lattices on one extreme,
   - something similar to Boolean algebras on the other.
2. We provide a duality for these structures, which specializes in two directions: (essentially) to Stone duality on one extreme and to Yosida duality on the other.

A hint: both sides have "addition"

In Yosida duality, $C(X, \mathbb{R})$ has a genuine addition $f + g$, defined everywhere.

In a Boolean algebra, $x \vee y$ behaves *like* an addition — but only when x and y are disjoint.

Idea: embed Boolean algebras into algebraic structures where a *true* addition is always defined.

Example:

$$\{0, 1\} \hookrightarrow \mathbb{Z}.$$

One algebraic structure to unify them

A good abstraction is: **lattice-ordered Abelian group** (*Abelian ℓ–group*) (1940):

- an Abelian group $(G, +, 0, -)$,
- equipped with a lattice order $(\vee, \wedge)$,
- that is translation–invariant $(x \leq y \Rightarrow x + z \leq y + z)$.

These objects can specialize in two directions:

- if highly divisible $\Rightarrow$ they look like vector lattices,
- if poorly divisible $\Rightarrow$ they look like Boolean algebras.

(Connection to MV-algebras (Łukasiewicz many-valued propositional logic).)

Definition

A *metrically complete unital Abelian ℓ-group* is a lattice-ordered Abelian group G equipped with a designated element $1 \in G$ such that the following defines a complete metric on G:

$$\mathrm{dist}(v, w) \coloneqq \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \geq 0, q > 0, \text{ and } p(-1) \leq q(v - w) \leq p1 \right\}.$$

Example

Given a compact Hausdorff space X:

$$C(X, \mathbb{R}) \coloneqq \{f \colon X \to \mathbb{R} \mid f \text{ is continuous}\},$$

$$C\left(X, \frac{1}{n}\mathbb{Z}\right) \coloneqq \left\{f \colon X \to \frac{1}{n}\mathbb{Z} \mid f \text{ is continuous}\right\}.$$

We can also mix the values that the functions can take at different points...

Representation theorem, Goodearl & Handelman 1980

- Let X be a compact Hausdorff space. For each $x \in X$, let D_x be either $\mathbb{R}$ or $\frac{1}{n}\mathbb{Z}$ (for some $n \in \mathbb{N} \setminus \{0\}$). Then,

$$\left\{ f \colon X \to \mathbb{R} \mid f \text{ continuous, } \forall x \in X \ f(x) \in D_x \right\}$$

  is a metrically complete unital Abelian ℓ-group.
- Every metrically complete unital Abelian ℓ-group can be represented in this way.

Example

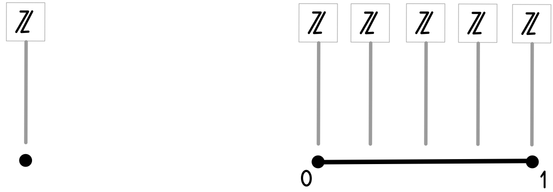
- If $D_x = \mathbb{R}$ for all $x \in X$, then we get $C(X, \mathbb{R})$.
- If $D_x = \mathbb{Z}$ for all $x \in X$, then we get $C(X, \mathbb{Z})$.

Our aim

Our aim (A., Marra, Spada): make the Goodearl-Handelman representation into a **categorical duality**, so that we have a representation of morphisms, and that we can transfer all problems that can be expressed in the categorical language.

Not a 1:1 correspondence

Goodearl & Handelman's representation is not a 1:1 correspondence.



In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

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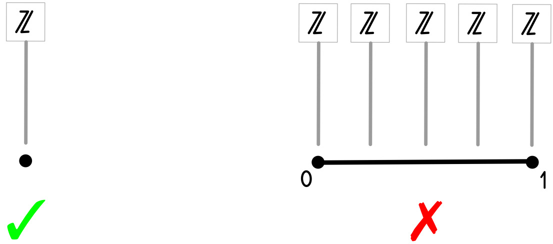
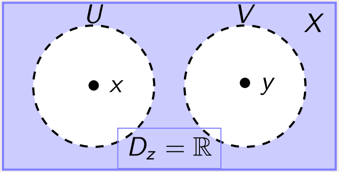


In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

Remedy: Separation

For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



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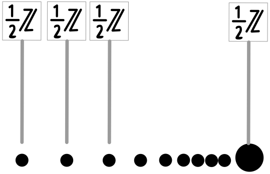

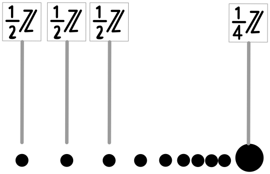

In both cases

$$\left\{ f\colon X \to \mathbb{R} \mid f \text{ cont. and, for all } x \in X,\ f(x) \in D_x \right\} =$$

$$= \left\{ f\colon X \to \frac{1}{2}\mathbb{Z} \mid f \text{ is eventually constant} \right\}.$$

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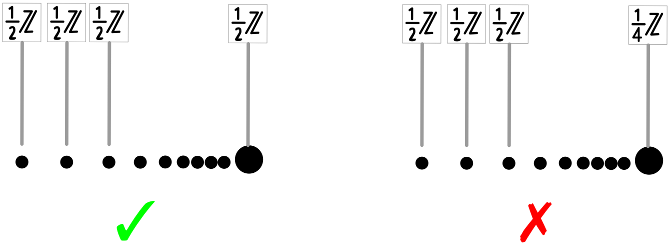

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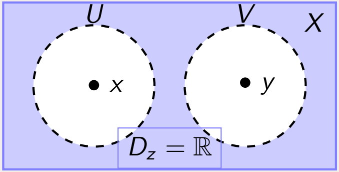
Remedy: Compactness

For every $A \in \{\mathbb{R}\} \cup \left\{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\right\}$, the set $\{x \in X \mid D_x \subseteq A\}$ is compact.

Definition

A *normal a-space* (for "arithmetically normal arithmetic space") is a topological space X equipped with a family $(D_x)_{x \in X}$ with $D_x \in \{\mathbb{R}\} \cup \left\{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\right\}$ such that

1. (Separation) For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



2. (Compactness) For every $A \in \{\mathbb{R}\} \cup \left\{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\right\}$, the set $\{x \in X \mid D_x \subseteq A\}$ is compact.

Example: $[0,1]$ with $D_{\frac{p}{q}} = \frac{1}{q}\mathbb{Z}$ and $D_x = \mathbb{R}$ otherwise.

A new duality (A., Marra, Spada, 2025)

The category of **metrically complete unital Abelian ℓ-groups** (and homomorphisms) is dually equivalent to the category of **normal a-spaces** (and continuous “denominator-decreasing” maps).

M. Abbadini, V. Marra, L. Spada.
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Recovering Stone and Yosida dualities

Stone: $D_x = \mathbb{Z}$ for every x

The separation axiom says that X is a **Stone space**.

$$X \longmapsto C(X, \mathbb{Z}).$$

Taking the unit interval (Mundici's functor) gives a **Boolean algebra**:

$$\{x \in C(X, \mathbb{Z}) \mid 0 \leq x \leq 1\} = C(X, \{0,1\}) \cong \mathrm{Clop}(X).$$

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Yosida: $D_x = \mathbb{R}$ for every x

Every **compact Hausdorff space** X is allowed.

$$X \longmapsto C(X, \mathbb{R}),$$

a metrically complete unital vector lattice.

Translating problems, again

Do metrically complete unital Abelian ℓ-groups have the **amalgamation property**?

Algebra (amalgamation).

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & B \arrow[d, hook, dashed, "u"] \\ C \arrow[r, hook, dashed, "v"'] & D \end{tikzcd}$$

Topology (normal a-spaces).

$$\begin{tikzcd} P \arrow[r, two heads, dashed, "p"] \arrow[d, two heads, dashed, "q"'] & X \arrow[d, two heads, "f"] \\ Y \arrow[r, two heads, "g"'] & Z \end{tikzcd}$$

A concluding reflection

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Baez, Dolan (2001):

> *"... an equation is only interesting or useful to the extent that the two sides are different!"*

The equation

$$e^{i\omega} = \cos(\omega) + i\sin(\omega)$$

is far more interesting and useful than

$$1 = 1,$$

precisely because its two sides have such different shapes.

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Thank you!